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..vell '05

NEW PLANE AND SOLID GEOMETRY

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PREFACE.

In presenting a revision of their "Plane and Solid Geometry" (Boston, 1895), the authors feel that an explanation of its distinctive features may be of service to the teacher.

It is sometimes asserted that we should break away from the formal proofs of Euclid and Legendre and lead the student to independent discovery, and so we find text-books that give no proofs, others that give hints of the demonstrations, and still others that draw out the demonstration by a series of questions which, being capable of answer in only one way, merely conceal the Euclidean proof. But, after all, the experience of the world has been that the best results are secured by setting forth a minimum of formal proofs as models, and a maximum of unsolved or unproved propositions as exercises. This plan has been followed by the authors, and the success of the first edition has abundantly justified their action.

There is a growing belief among teachers that such of the notions of modern geometry as materially simplify the ancient should find place in our elementary text-books. With this belief the authors are entirely in sympathy. Accordingly they have not hesitated to introduce the ideas of one-to-one correspondence, of anti-parallels, of negative magnitudes, of general figures, of prismatic space, of similarity of point systems, and such other concepts as are of real value in the early study of the science. All this has been done in a con-

servative way, and such material as the first edition showed to be at all questionable has been omitted from the present revision.

Within comparatively recent years the question of methods of attack has interested several leading writers. Whatever has been found to be usable in elementary work the authors have inserted where it will prove of most value. To allow the student to grope in the dark in his efforts to discover a proof, is such a pedagogical mistake that this innovation in American text-books has been generally welcomed. Upon this point the authors have freely drawn from the works of Petersen of Denmark, and of Rouché and de Comberousse of France, and from the excellent treatise recently published by Hadamard (Paris, 1898).

With this introduction of modern concepts has necessarily come the use of certain terms and symbols which may not generally be recognized by teachers. These have, however, been chosen only after most conservative thought. None is new in the mathematical world, and all are recognized by the leading writers of the present time. They certainly deserve place in our elementary treatises on the ground of exactness, of simplicity, and of their general usage in mathematical literature.

The historical notes of the first edition have been retained, it being the general consensus of opinion that they add materially to the interest in the work. For teachers who desire a brief but scholarly treatment of the subject the authors refer to their translation of Fink's "History of Elementary Mathematics" (Chicago, The Open Court Publishing Co., 1899). For the limitations of elementary geometry, the impossibility of trisecting an angle, squaring a circle, etc., teachers should read the authors' translation of

Klein's valuable work, "Famous Problems of Elementary Geometry" (Boston, Ginn & Company).

It is impossible to make complete acknowledgment of the helps that have been used. The leading European text-books have been constantly at hand. Special reference, however, is due to such standard works as those of Henrici and Treutlein, "Lehrbuch der Elementar-Geometrie," the French writers already mentioned, and the noteworthy contributions of the recent Italian school represented by Faifofer, by Socci and Tolomei, and by Lazzeri and Bassani.

Teachers are urged to consider the following suggestions in using the book:

1. Make haste slowly at the beginning of plane and of solid geometry.

2. Never attempt to give all of the exercises to any class. Two or three hundred, selected by the teacher, should suffice.

3. Require frequent written work, thus training the eye, the hand, and the logical faculty together. The authors' Geometry Tablet (Ginn & Company) is recommended for this work.

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June 15, 1899.

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PLANE AND SOLID GEOMETRY.

PLANE GEOMETRY.

INTRODUCTION.

I. ELEMENTARY DEFINITIONS.

1. In **Arithmetic** the student has considered the science of *numbers*, and has found, for example, that a number which ends in 5 or 0 is divisible by 5.

In **Algebra** he has studied, among other things, the *equation*, and has found that if $\frac{1}{2}x - 1 = 5$, x must equal 12.

In **Geometry** he is to study *form*, and he will find, for example, that two triangles must necessarily be equal if the three sides of the one are respectively equal to the three sides of the other.

Before beginning the subject, however, there are certain terms which, although familiar, are used with such exactness as to require careful explanation. These terms are *solid, surface, line, angle* (with various kinds of each), and *point*. As with most elementary mathematical terms, such as *number, space*, etc., it is difficult to give them simple and satisfactory definition. Explanations can, however, be given which will lead the student to a reasonable understanding of them.

2. The space with which we are familiar and in which we live is evidently divisible. Any limited portion of space is called a **solid**.

In geometry no attention is given to the substance of which the solid is composed. It may be water, or iron, or air, or

wood, or it may be a vacuum. Indeed, geometry considers only the *space occupied* by the substance. This space is called a *geometric solid,* or simply a *solid,* while the substance is called a *physical solid.* Thus, a ball is a physical solid; the space which the ball occupies is a geometric solid.

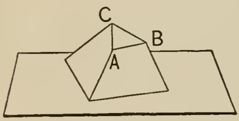
3. That which separates one part of space from an adjoining part is called a **surface.** So we speak of the surface of a ball, the surface of the earth, etc.

4. Every surface is divisible. That which separates one part of a surface from an adjoining part is called a **line.**

5. Every line is divisible. That which separates one part of a line from an adjoining part is called a **point.**

A point is not divisible.

Thus, in the figure the surface of the block separates the space occupied by the block from all the rest of space. This surface is divisible in many ways; for example, it is divided into two parts by the line passing from A through B and C and back to A. This line is divisible in many ways; for example, it is separated into three parts by the points A, B, C. In the case of a line that returns into itself, — *i.e.* a *closed* line, like the one just mentioned, — *two* points are necessary completely to separate one part from the other.



It is impossible to draw mechanically a geometric line. A chalk mark, a thread, a fine wire, an ink mark, are all very thin physical solids used to *represent* lines; for this purpose they are very helpful. So, too, a dot may be used to *represent* a point, and a sheet of paper may be used to *represent* a surface, although each is really a physical solid.

6. The preceding definitions start from the *solid* and take the *surface, line,* and *point* in order. It is also possible to start with the *point* and proceed in reverse order.

The **point** is the simplest geometric concept; it has position, but not magnitude.

A moving point describes a **line.**

This may be *represented* by a pencil point moving on a piece of paper.

A moving line describes, in general, a **surface.**

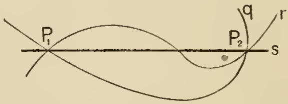
This may be *represented* by a crayon lying flat against the blackboard, and moving sidewise. How may a line move so as not to describe a surface ?

A moving surface describes, in general, a **solid.**

Thus, the surface of a glass of water, as it moves upward, may be said to describe a solid. How may a surface move so as not to describe a solid ?

7. Through two points any number of lines may be imagined to pass.

For example, through the points P_1, P_2 (read "P-one, P-two") the lines q, r, s may be imagined to pass.



A **straight line** is a line which is determined by any two of its points.

In the figure, s represents a straight line, for, given the points P_1, P_2 on the line, its position is fixed; it is *determined.*

But q and r do not represent straight lines, because P_1 and P_2 do not determine them.

The word *line*, used alone, is to be understood to refer to a straight line.

The expression *straight line* is used to mean both an unlimited straight line and a portion of such a line. In case of doubt, *line-segment*, or merely *segment*, is used to mean a limited straight line.

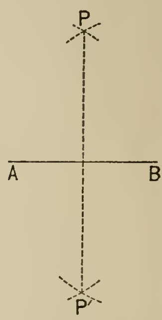
As has been seen, a point is usually named by some capital letter. A segment is usually named by naming its end points, or by a single small letter.

In the annexed figure, AB, AC, BC, and o are marked off.

Two segments are said to be *equal* when they can be made to coincide.

8. If three points, A, B, C, are taken in order on a line, as in the preceding figure, then the line-segment AC is called the **sum** of the line-segments AB and BC, and AB is called the **difference** between AC and BC.

9. If a point divides a line-segment into two equal segments, it is said to **bisect** the line-segment and is called its **mid-point**.



A line is easily bisected by the use of a straight-edge and compasses, thus:

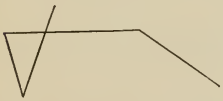
With centers A and B, and equal radii, describe arcs intersecting at P and P'.

Draw PP'. This bisects AB.

The proof of this fact is given later.

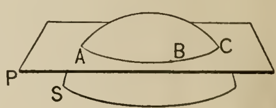
10. If a segment is drawn out to greater length, it is said to be **produced.**

To produce AB means to extend it through B, toward C, in the second figure in § 7. To produce BA means to extend it through A, away from B.



11. A line not straight, but made up of straight lines, is called a **broken line.**

12. Through three points, not in a straight line, any number of surfaces may be imagined to pass.



For example, through the points A, B, C the surfaces P and S may be imagined to pass.

A **plane surface** (also called a *plane*) is a surface which is determined by any three of its points not in a straight line.

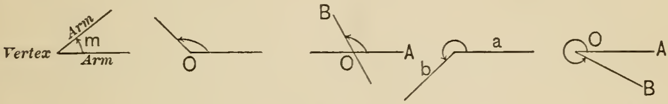
In the figure, P represents a plane, for it is *determined* by the points A, B, C. But S does not represent such a surface.

A plane is indefinite in extent unless the contrary is stated. To *produce* it means to extend it in length or breadth.

13. If two lines proceed from a point, they are said to form an **angle,** the lines being called the **arms,** and the point the **vertex,** of that angle.

The *size of the angle* is independent of the length of the arms; the size depends merely upon the amount of turning necessary to pass from one arm to the other.

The methods of naming an angle will be seen from the annexed figures. It is convenient to letter an angle around the vertex, as indicated by the arrows, that is, opposite to the course of clock-hands, or *counter-clockwise.*

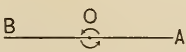


Angle m. Angle O. Angle AOB. Angle ab. Angle AOB.

A line proceeding from the vertex, turning about it counter-clockwise from the first arm to the second, is said to *turn through the angle, the angle being greater as the amount of turning is greater.*

14. If the two arms of an angle lie in the same straight line on opposite sides of the vertex, a **straight angle** is said to be formed. If the angle still further increases, until the moving arm has performed a complete revolution, thus passing through two straight angles, a **perigon** is said to be formed.

For practical purposes angles are measured in degrees, minutes, and seconds. A perigon is said to contain 360°.



AOB, a straight angle.
BOA, a straight angle.

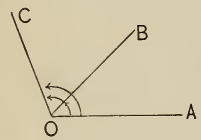
A perigon, or angle of 360°.

In general, if two lines are drawn from O, two angles, each less than a perigon, are formed. Of these the smaller is always to be understood if "the angle at O" is mentioned, unless the contrary is stated.

15. If a line turns through an angle, all points or line-segments through which it passes in its turning, except the vertex, are said to be *within the angle.* Other points or lines are either on the arms or *without the angle.*

16. Two angles, ab, $a'b'$, are said to be *equal* when, without changing the relative position of a and b, angle ab may be placed so that a lies along a', and b along b'.

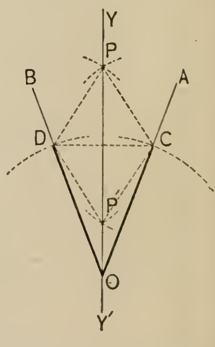
This equality is tested by placing one angle on the other, the vertices coinciding. Then if the arms can be made to coincide, the angles are equal, otherwise not.



17. If three lines, OA, OB, OC, proceed from a common point O, OB lying within the angle AOC, then angles AOB and BOC are called **adjacent angles.** Angle AOC is called the *sum of the angles* AOB, BOC. Either of the adjacent angles is called the *difference* between angle AOC and the other of the adjacent angles.

As two angles may be added, so several may be added.

18. If a line divides an angle into two equal angles, it is said to *bisect* the angle and is called its *bisector.*



In the annexed figure, if angle AOY equals angle YOB, then OY is the bisector of angle AOB.

And, in general, to *bisect* any magnitude means to divide it into two equal parts.

An angle is easily bisected by the use of a straight-edge and compasses, thus:

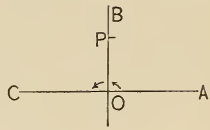
If AOB is the given angle, mark off with the compasses OC equal to OD.

Then with C and D as centers and CD as a radius draw two arcs intersecting at P and P'.

The line joining P or P' with O is the required bisector. The proof of this fact is given later.

19. A **right angle** is half of a straight angle.

It follows from this definition that *the sum of two right angles is a straight angle;* and from the definitions of a straight angle and of a perigon, *that the sum of two straight angles, or of four right angles, is a perigon.*



It also follows that a straight angle contains 180° and a right angle contains 90°.

20. If two lines meet and form a right angle, each line is said to be **perpendicular** to the other.

Each is also spoken of as *a* perpendicular to the other. Thus, in the preceding figure, BO is perpendicular to CA, or is *a* perpendicular to CA. The segment PO is called *the* perpendicular from P to CA, since it will presently be proved that it is unique; that is, that there is one and only one perpendicular. O is called the *foot* of that perpendicular.

The word *unique*, meaning *one and only one*, is frequently used in mathematics.

A line is easily drawn perpendicular to another line by the use of a straight-edge and compasses. This is seen in the figure in § 9, where PP' is perpendicular to AB.

21. An angle less than a right angle is said to be **acute**; one greater than a right angle but less than a straight angle is said to be **obtuse**; one greater than a straight angle but less than a perigon is said to be **reflex** or **convex.**

22. Two lines which form an acute, obtuse, or reflex angle are said to be **oblique** to each other.

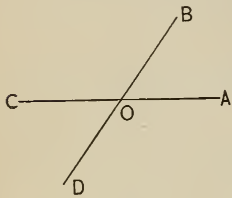
Acute, obtuse, and reflex angles are classed under the general term *oblique angles.*

The meaning of the expressions *oblique lines, an* oblique, *foot* of an oblique, will be understood from § 20.

Draw a figure representing acute, obtuse, and reflex angles, oblique lines, an oblique from P to CA, the foot of an oblique.

23. Two angles are said to be **complements** of each other if their sum is a right angle. Two angles are said to be **supplements** of each other if their sum is a straight angle. Two angles are said to be **conjugates** of each other if their sum is a perigon.

If one angle is the complement of another, the two angles are said to be *complemental* or *complementary*. Similarly, if one angle is the supplement of another, the two angles are said to be *supplemental* or *supplementary*.



In the annexed figure, angles AOB and BOC are supplemental, also angles BOC and COD, etc.

24. If two lines, CA, DB, intersect at O, as in the above figure, the angles AOB and COD are called **vertical** or **opposite** angles; also the angles BOC and DOA.

Exercises. 1. How many degrees in a right angle? How many minutes? How many seconds?

2. What is the complement of one-half of a right angle? of one-fourth?

3. How many degrees in the supplement of an angle of (a) 75°? (b) 90°? (c) 150°? (d) 179°?

4. Also in the complement of an angle of (a) 75°? (b) 1°? (c) 89°? (d) 45°? (e) 90°? (f) 0°?

5. Also in the conjugate of an angle of (a) 270°? (b) 180°? (c) 359°? (d) 90°? (e) 1°? (f) 360°?

6. Draw a figure showing that two straight lines determine one point; also one showing that three straight lines determine, in general, three points.

7. How many degrees in each of the two conjugate angles which the hour and minute hands of a clock form at 4 o'clock?

8. If six lines, proceeding from a point, divide a perigon into six equal angles, express one of those angles (a) in degrees, (b) as a fraction of a right angle, (c) as a fraction of a straight angle.

2. THE DEMONSTRATIONS OF GEOMETRY.

25. The **object of geometry** is the investigation of *truths* concerning combinations of lines and points, and of the methods of making certain *constructions* from lines and points.

26. A **proposition** is a statement of either a truth to be demonstrated or a construction to be made.

For example, geometry investigates this proposition: If two lines intersect, the vertical angles are equal. It also investigates the methods of drawing a line perpendicular to another line, and various other propositions requiring some construction.

Propositions are divided into two classes — *theorems* and *problems*.

A **theorem** is a statement of a geometric *truth* to be demonstrated.

A **problem** is a statement of a geometric *construction* to be made.

For example: THEOREM, *If two lines intersect, the vertical angles are equal.* — PROBLEM, *Required through a point in a line to draw a perpendicular to that line.*

27. There are a few *geometric* statements so obvious that the truth of them may be taken for granted, and a few geometric operations so simple that it may be assumed that they can be performed. Such a statement, or the claim to perform such an operation, is called a **postulate.**

The geometric operations thus assumed require the use of the straight-edge and compasses. *The straight-edge and the compasses are the only instruments recognized in elementary geometry.*

The postulates used in this work are set forth from time to time as required. At present three general classes suffice, as follows:

28. Postulates of the Straight Line.

1. *Two points determine a straight line.*

This follows from the definition.

2. *Two straight lines in a plane determine a point.*

3. *A straight line may be drawn and revolved about one of its points as a center so as to include any assigned point in space.*

4. *A straight line-segment may be produced.*

5. *A straight line is divided into two parts by any one of its points.*

29. Postulates of the Plane.

1. *Three points not in a straight line determine a plane.*

This follows from the definition.

2. *A straight line through two points in a plane lies wholly in the plane.*

Thus, if part of a straight line lies in an unlimited plane blackboard, the whole line lies in the blackboard.

3. *A plane may be passed through a straight line and revolved about it so as to include any assigned point in space.*

4. *A portion of a plane may be produced.*

5. *A plane is divided into two parts by any one of its straight lines, and space is divided into two parts by any plane.*

30. Postulate of Angles.

All straight angles are equal.

31. There are also a number of simple statements, *of a general nature*, so obvious that the truth of them may be taken for granted. These are called **axioms**.

The following are the axioms most frequently used in geometry, and they are so important that they should be learned by number.

32. Axioms.

1. *Things which are equal to the same thing, or to equal things, are equal to each other.*

That is, (1) if $A = B$, and $C = B$, then $A = C$. Or, (2) if $A = B$, and $B = C$, and $C = D$, then $A = D$.

2. *If equals are added to equals, the sums are equal.*

That is, if $A = B$, and if $C = D$, then $A + C = B + D$.

3. *If equals are subtracted from equals, the remainders are equal.*

That is, if $A = B$, and if $C = D$, then $A - C = B - D$.

4. *If equals are added to unequals, the sums are unequal in the same sense.*

That is, if $A = B$, and if C is greater than D, then $A + C$ is greater than $B + D$.

5. *If equals are subtracted from unequals, the remainders are unequal in the same sense.*

That is, if $A = B$, and if C is greater than D, then $C - A$ is greater than $D - B$.

6. *If equals are multiplied by equals, the products are equal.*

That is, if $A = B$, and m is any number, then $mA = mB$.

7. *If equals are divided by equals, the quotients are equal.*

That is, as in axiom 6, $\frac{A}{m} = \frac{B}{m}$. It will be seen that axiom 6 covers axiom 7, for m may be a fraction.

8. *The whole is greater than any of its parts, and equals the sum of all its parts.*

The latter part of this axiom is merely the definition of *whole*.

9. *If three magnitudes are so related that the first is greater than the second, while the second is greater than, or equal to, the third, then the first is greater than the third.*

E.g. if A is greater than B, and if B is greater than, or equal to, C, then A is greater than C.

33. SYMBOLS AND ABBREVIATIONS.

The following are used in this work, and are inserted here merely for reference, and not for memorizing:

e.g.	Latin, *exempli gratia*, for example.
i.e.	Latin, *id est*, that is.
$\because$	since.
$\therefore$	therefore.
pt., pts.	point, points.
rt.	right.
st.	straight.
ax.	axiom.
post.	postulate.
def.	definition.
prop.	proposition.
th.	theorem.
pr.	problem.
cor.	corollary.
subst.	substitution.
prel.	preliminary.
const.	construction.
ppd.	parallelepiped.
$\frown$	arc.
⊙, Ⓢ	circle, circles.
$\triangle$, $\triangle$s	triangle, triangles.
□, ⊡	square, squares.
▭, ▭s	rectangle, rectangles.
▱, ▱s	parallelogram, parallelograms.
$\angle$, $\angle$s	angle, angles.
$+$	plus, increased by.
$-$	minus, diminished by.
$\times$, $\cdot$,	and absence of sign, denote multiplication.
$\div$, /, :,	and fractional form, denote division.
$=$	is equal, or equivalent, to.
$\equiv$	is identical with, as $AB \equiv AB$, or coincides with.
$\cong$	is congruent to.
$\backsim$	is similar to.
$\doteq$	approaches as a limit.
$>$	is greater than.
$<$	is less than.
$\neq$	is not equal to, *i.e.* $>$ or $<$.
$\not>$	is not greater than, *i.e.* $=$ or $<$.
$\not<$	is not less than, *i.e.* $=$ or $>$.
$\perp$	is perpendicular to, or a perpendicular.
$\parallel$	is parallel to, or a parallel.
.....	and so on.

The above take the plural also; thus, $=$ means *are* equal, as well as *is* equal.

The manner of reading some of the familiar symbols is suggested, as follows:

P', P-prime; P'', P-second; P''', P-third, etc.
P_1, P-one; P_2, P-two, etc.
$A'B'$, A-prime B-prime, etc.
A_1', A-one-prime, etc.

References to preceding propositions are made by book and proposition thus, I, prop. IV; if the first Roman numeral is omitted, the proposition is in the current book. Section references are also used.

Other simple abbreviations are occasionally used, but they will be easily understood.

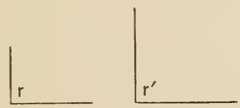
3. PRELIMINARY PROPOSITIONS.

34. The following theorems are designed to show to the beginner the nature of a geometric proof, and to lead him by easy steps to appreciate the logic of geometry. Some of them might properly have been incorporated in Book I, and others might have been omitted altogether; but they form a group of simple propositions which lead the student up to the more difficult work of geometry, and for that reason they are inserted here. The student and the teacher are advised to proceed slowly until the logic of the subject is understood, and under no circumstances to allow mere memorizing of the proofs.

Proposition I.

35. Theorem. *All right angles are equal.*

Suggestion. The only angles of whose equality we are thus far assured are straight angles. Hence in some way we must base our proof of this theorem on the postulate of angles, which asserts this fact. We then consider how a right angle is related to a straight angle, and the proof is at once suggested.



Given any two right angles, r, r'.

To prove that $r = r'$.

Proof. 1. r and r are halves of straight angles. Def. rt. ∠

(§ 19. A right angle is half of a straight angle.)

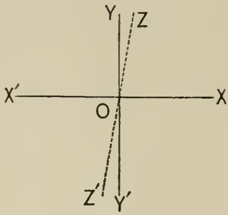
2. All straight angles are equal. § 30

3. ∴ all right angles, and hence r and r', are equal. Ax. 7

(If equals are divided by equals, the quotients are equal.)

PROPOSITION II.

36. Theorem. *At a given point in a given line not more than one perpendicular can be drawn to that line in the same plane.*



Given $YY' \perp XX'$ at O.

To prove that no other perpendicular can be drawn to XX', at O, in the same plane.

Proof. 1. Suppose that another $\perp$, ZZ', could be drawn.

2. Then $\angle XOZ$ would be a rt. $\angle$. Def. $\perp$

(If two lines meet and form a rt. $\angle$, each is said to be $\perp$ to the other.)

3. But $\angle XOY$ is a rt. $\angle$. Given; def. $\perp$, § 20

(For it is given that $YY' \perp XX'$, and the def. of a $\perp$ is given in step 2.)

4. $\therefore \angle XOY$ would equal $\angle XOZ$. Prop. I

(All right angles are equal.)

5. But this is impossible. Ax. 8

(The whole is greater than any of its parts, etc.)

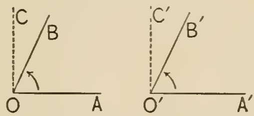
6. $\therefore$ the supposition of step 1 is absurd, and a second perpendicular is impossible.

NOTE. In prop. I we proved directly from the definition of straight angle that all right angles are equal. In prop. II a different method of proof is followed. We have here supposed that the theorem is false and have shown that this supposition is absurd. Such proofs have long been known by the name "*reductio ad absurdum*," a reduction to an absurdity. They are also called *indirect* proofs.

PROPOSITION III.

37. Theorem. *The complements of equal angles are equal.*

SUGGESTION. Three lines of proof may present themselves. We may base our proof on the equality of straight angles, as we did in prop. I, or we may take an indirect proof as in prop. II, beginning by supposing the theorem false and showing the absurdity of this supposition, or we may base the proof on prop. I. Since the complements suggest right angles, which of the three methods would it probably be best to follow?



Given two equal $\angle$s, AOB, $A'O'B'$, and their complements, BOC, $B'O'C'$, respectively.

To prove that $\angle BOC = \angle B'O'C'$.

Proof. 1. $\angle$s AOC and $A'O'C'$ are rt. $\angle$s. Def. compl.

(§ 23. Two $\angle$s are said to be complements if their sum is a rt. $\angle$.)

2. $\therefore \angle AOC = \angle A'O'C'$. Prop. I

(All right angles are equal.)

3. But $\angle AOB = \angle A'O'B'$. Given

4. $\therefore \angle BOC = \angle B'O'C'$. Ax. 3

(If equals are subtracted from equals, the remainders are equal.)

PROPOSITION IV.

38. Theorem. *The supplements of equal angles are equal.*

Let the student draw the figure and give the proof after the manner of prop. III. Use only four steps in the proof.

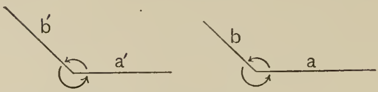
Given

To prove

Proof.

PROPOSITION V.

39. Theorem. *The conjugates of equal angles are equal.*



Given two equal angles, ab, $a'b'$.

To prove that $\angle ba = \angle b'a'$.

Proof. 1. The given ∠s may be so placed that a lies along a', and b along b'. Def. equal ∠s

(§ 16. Two ∠s, ab, $a'b'$, are said to be equal when $\angle ab$ can be placed so that a lies along a', and b along b'.)

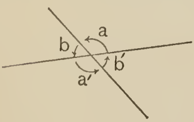
2. But then $\angle ba$ must equal $\angle b'a'$. Def. equal ∠s

---

PROPOSITION VI.

40. Theorem. *If two lines cut each other, the vertical angles are equal.*

SUGGESTION. After examining the figure the student might say that because $\angle a + \angle b =$ st. ∠, and $\angle b + \angle a' =$ st. ∠, $\therefore \angle a + \angle b = \angle b + \angle a'$, and then subtract $\angle b$ from these equals; or he might say that $\angle a = \angle a'$ because each is the supplement of $\angle b$. He should always feel encouraged to try various proofs, selecting the shortest and the clearest. Does the following proof meet these requirements?



Given two lines cutting each other, forming two pairs of opposite angles, a, a', and b, b'.

To prove that $\angle a = \angle a'$.

Proof. 1. $\angle a$ and $\angle a'$ are supplements of $\angle b$. Def. suppl.

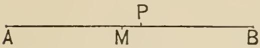
(§ 23. Two ∠s are said to be supplements if their sum is a st. ∠.)

2. $\therefore \angle a = \angle a'$. Prop. IV

(The supplements of equal angles are equal.)

PROPOSITION VII.

41. Theorem. *A line-segment can be bisected in only one point.*



Given a line-segment AB, bisected at M.

To prove that there is no other point of bisection.

Proof. 1. Suppose another point of bisection exists, as P, between M and B.

2. Then since AM and AP are both halves of AB, they are equal. Ax. 7

(State ax. 7.)

3. But this is impossible, for AM is part of AP. Ax. 8

(State ax. 8.)

4. ∴ the supposition that there is a second point of bisection is absurd.

(Another *reductio ad absurdum*, as in prop. II.)

PROPOSITION VIII.

42. Theorem. *An angle can be bisected by only one line.*

(The student may prove this after the manner of prop. VII.)

Exercises. 9. Of two supplemental angles, a and b, (a) suppose $a = 2b$, how many degrees in each? (b) suppose $a = 3b$, how many?

10. How many straight lines are, in general, determined by three points? by four? (The points in the same plane.)

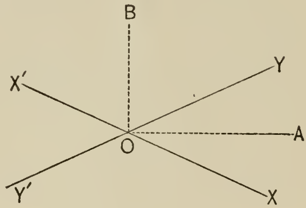
11. If of five angles, a, b, c, d, e, whose sum is a perigon, $a = 20°$, $b = 30°$, $c = 40°$, $d = 50°$, how many degrees in e?

12. Of three angles whose sum is a perigon, the first is twice the second, and the second three times the third; how many degrees in each?

PROPOSITION IX.

43. Theorem. *The bisectors of two adjacent angles formed by one line cutting another are perpendicular to each other.*

SUGGESTION. Considering the figure, we see that to prove $OA \perp OB$ we must show that $\angle AOB$ is a rt. $\angle$. Now the only way that we have as yet of showing an angle to be a right angle is to show that it is half of a straight angle. But evidently $\angle AOY$ is half of $\angle XOY$, because $\angle XOY$ is bisected; similarly, $\angle YOB$ is half of $\angle YOX'$, and this suggests the following proof.



Given two lines, XX', YY', cutting at O; also OA, OB, bisecting $\measuredangle s\ XOY$, YOX', respectively.

To prove that $OA \perp OB$.

Proof. 1. $\angle AOY = \frac{1}{2} \angle XOY$. Given; § 18

2. $\angle YOB = \frac{1}{2} \angle YOX'$. Given; § 18

3. $\therefore \angle AOB = \frac{1}{2} \angle XOX'$. Ax. 2

(If equals are added to equals, the sums are equal.)

4. $\therefore \angle AOB = \frac{1}{2}$ of a st. $\angle$. Def. st. $\angle$

(§ 14. If the two arms of an $\angle$ lie in the same st. line on opposite sides of the vertex, a st. $\angle$ is said to be formed.)

5. $\therefore \angle AOB =$ a rt. $\angle$. Def. rt. $\angle$

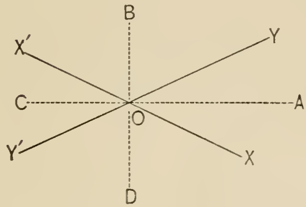
(§ 19. A rt. $\angle$ is half of a st. $\angle$.)

6. $\therefore OA \perp OB$. Def. $\perp$

(§ 20. If two lines meet and form a rt. $\angle$, each line is said to be $\perp$ to the other.)

PROPOSITION X.

44. Theorem. *The bisectors of the four angles which two intersecting lines make with each other form two straight lines.*



Given XX' intersecting YY' at O, OA bisecting $\angle XOY$, OB bisecting $\angle YOX'$, OC bisecting $\angle X'OY'$ and OD bisecting $\angle Y'OX$.

To prove that COA and DOB are straight lines.

Proof. 1. $\measuredangle s$ AOB and BOC are rt. $\measuredangle s$. — Prop. IX

(State prop. IX.)

2. $\therefore$ the two together form a st. angle. — Def. rt. $\angle$

(§ 19. State the definition.)

3. $\therefore$ COA is a st. line. — Def. st. $\angle$

(§ 14. State the definition.)

4. Similarly for DOB.

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45. The nature of a logical proof should now be understood. Before continuing, however, the following points should be emphasized:

a. Every statement in a proof must be based upon a postulate, an axiom, a definition, or some proposition previously considered of which the student is prepared to give the proof again when he refers to it.

b. No statement is true simply because it appears to be true from a figure which the student may have drawn, no matter how carefully. Many cases will be found, for example, where angles appear equal when they are not so.

c. The arrangement of the discussion of a theorem is as follows:

GIVEN. Here is stated, with reference to the figure which accompanies the proof, whatever is given by the theorem.

TO PROVE. Here is stated the exact conclusion to be derived from what is given.

PROOF. Here are set forth, in concise steps, the statements to prove the conclusion just asserted. If the proof is written on the blackboard, the steps should be numbered for convenient reference by class and teacher. The teacher will state how much in the way of written or indicated authorities shall be required after each step.

COROLLARY. A **corollary** is a proposition so connected with another as not to require separate treatment. The proof is usually simple, but it must be given with the same accuracy as that of the proposition to which it is attached. It is usually sufficient to say, This is proved in step 4; or, This follows from steps 2 and 5 by axiom 3, etc. In every case the student should (1) clearly prove the corollary, but (2) do so as concisely as possible. A corollary may also follow from a definition; thus, from the definitions of *Proposition* and *Theorem* the following might be stated as a corollary: Every theorem is a proposition, but not every proposition is a theorem; and as a part of our definition of a *Perigon* we incorporated the corollary (the term then being undefined) that a perigon equals two straight angles.

NOTE. Any item of interest may be inserted under this head.

Exercises. 13. Of the proofs of the preliminary theorems, state which are direct and which indirect. (See note on p. 14.)

14. How can you form a right angle by paper folding? Prove it.

BOOK I.—RECTILINEAR FIGURES.

1. TRIANGLES.

46. A **figure** is any combination of lines and points formed under given conditions.

E.g. an angle is a figure, for it is a combination of two lines and one point formed under the condition that the two lines proceed from the point.

47. A **rectilinear figure** is a figure of which all the lines are straight.

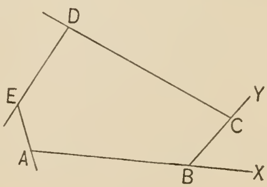
Plane geometry treats of figures in one plane,—*plane figures.*

Hence in plane geometry, which in this work extends through Books I to V inclusive, the word *figure* used alone denotes a *plane figure*, and all propositions and definitions refer to such figures placed in one plane.

48. If the two end-points of a broken line coincide, the figure obtained is called a **polygon**, and the broken line its **perimeter.** The vertices of the angles made by the segments of the perimeter are called the **vertices of the polygon,** and the segments between the vertices are called the **sides** of the polygon.

49. The perimeter of a polygon divides the plane into two parts, one finite (the part inclosed), the other infinite. The finite part is called *the surface of the polygon*, or for brevity simply *the polygon.*

A point is said to be *within* or *without* the polygon according as it lies within or without this finite part.



A polygon.

The figure $ABCDE$ is a polygon (the sides being produced for a subsequent definition).

50. In passing counter-clockwise around the perimeter of a polygon the angles on the left are called the **interior angles** of the polygon, or for brevity simply *the angles of the polygon.*

Such are the angles CBA, DCB, EDC, ... in the figure on p. 21.

51. If the sides of a polygon are produced in the same order, the angles between the sides produced and the following sides are called the **exterior angles** of the polygon.

Such are the angles XBC, YCD, ... in the figure on p. 21. They are the angles through which one would turn, at the successive corners, in walking around the polygon.

52. A line joining the vertices of any two angles of a polygon which have not a common arm, is called a **diagonal.**

Such a line would be the one joining A and C in the figure on p. 21. The sides, angles, and diagonals of a polygon are often called its *parts.*

53. A polygon which has all of its *sides* equal is called **equilateral.**

A polygon which has all of its *angles* equal is called **equiangular.**

54. Two polygons are said to be **mutually equilateral,** or one is said to be *equilateral* to the other, when the *sides* of the one are respectively equal to the *sides* of the other.

Two polygons are said to be **mutually equiangular,** or one is said to be *equiangular* to the other, when the *angles* of the one are respectively equal to the *angles* of the other.

55. A polygon of three sides is called a **triangle**; one of four sides, a **quadrilateral.**

56. Any side of a polygon may be called its **base,** the side on which the figure appears to stand being usually so called, as AB in the figure on p. 21.

In the case of a triangle, the vertex of the angle opposite the base is called the **vertex of the triangle,** the angle itself being called the *vertical angle of the triangle,* and the other two angles the *base angles.*

Thus, in the first triangle on p. 25, C is the vertex of the triangle, $\angle C$ is the vertical angle, $\angle A$ and $\angle B$ are the base angles.

57. Two figures which may be made to coincide in all their parts by being placed one upon the other are said to be **congruent.**

For example, two line-segments may be congruent, or two angles, or two triangles, etc.

58. The operation of placing one figure upon the other so that the two shall coincide is called *superposition,* and the figures are sometimes called *superposable* (a synonym of congruent).

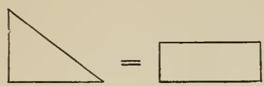
This is illustrated in prop. I.

Superposition is an imaginary operation. It is assumed as a postulate (§ 61) that figures may be moved about in space with no other change than that of position. The actual movement is, however, left for the imagination.

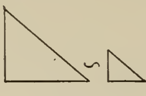
59. It will hereafter be explained and defined that polygons of the same shape are called *similar,* the symbol of similarity being $\backsim$, and that those of the same area are called *equal* or *equivalent,* the symbol being $=$. *Congruent figures are both similar and equal,* and hence the symbol for congruence is $\backsimeq$, a symbol used in modified form by the great mathematician Leibnitz.

The symbol $\backsim$ is derived from the letter S, the initial of the Latin *similis,* similar.

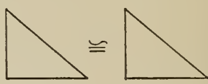
Many writers use *equal* for *congruent*, and *equivalent* for *equal*, as above defined. But because of the various meanings of the word *equal*, and its general use as a synonym for



Equality.



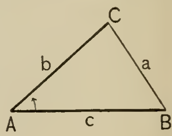
Similarity.



Congruence.

equivalent, the more exact word *congruent* with its suggestive symbol is coming to be employed. The student should be familiar with this other use of the words *equal* and *equivalent*.

60. It is customary to designate the sides of a triangle by the small letters corresponding to the capital letters which designate the opposite vertices.



Thus, in the figure, side a is opposite vertex A, etc.

61. It now becomes necessary to assume three other postulates.

Postulates of Motion.

1. *A figure may be moved about in space with no other change than that of position, and so that any one of its points may be made to coincide with any assigned point in space.*

That is, we may pick up one polygon and place it on another without changing its shape or size.

2. *A figure may be moved about in space while one of its points remains fixed.*

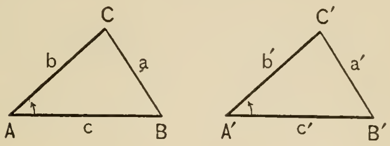
Such movement is called *rotation about a center*, the center being the fixed point.

3. *A figure may be moved about in space while two of its points remain fixed.*

Such movement is called *rotation about an axis*, the axis being the line determined by the two points.

PROPOSITION I.

62. Theorem. *If two triangles have two sides and the included angle of the one respectively equal to two sides and the included angle of the other, the triangles are congruent.*



Given the $\triangle$ ABC, $A'B'C'$ such that

$$c = c',$$
$$b = b', \text{ and}$$
$$\angle A = \angle A'.$$

To prove that $\triangle ABC \cong \triangle A'B'C'$.

Proof. 1. Place $\triangle A'B'C'$ on $\triangle ABC$ so that

A' falls on A, and § 61, 1

c' coincides with its equal c. § 61, 2

2. Then b' may be caused to fall on b,

because $\angle A' = \angle A$. Given; § 61, 3

3. Then C' will fall at C,

because $b' = b$. Given; § 57

4. $\therefore a'$ will coincide with a. § 28, 1

(Two points determine a straight line.)

5. $\therefore \triangle ABC \cong \triangle A'B'C'$, by definition of congruence. § 57

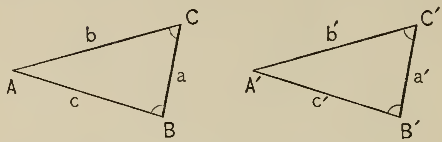
NOTES. This is a proof by superposition.

The theorem may be stated, A triangle is *determined* when two sides and the included angle are given.

In the exercises hereafter given, the proofs are to be given in full; when a question is asked, a proof of the answer is to be given; when a theorem is suggested, it is to be completely stated and then proved.

Proposition II.

63. Theorem. *If two triangles have two angles and the included side of the one respectively equal to two angles and the included side of the other, the triangles are congruent.*



Given the $\triangle ABC$ and $A'B'C'$ such that

$$\angle C = \angle C',$$
$$\angle B = \angle B', \text{ and}$$
$$a = a'.$$

To prove that $\triangle ABC \cong \triangle A'B'C'$.

Proof. 1. Place $\triangle A'B'C'$ on $\triangle ABC$ so that a' falls on a and $\angle C'$ coincides with its equal $\angle C$. § 61

2. Then B' will fall on B because $a' = a$. Given

3. Then c' will fall on c because $\angle B' = \angle B$. Given

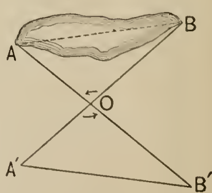
4. $\therefore A'$ will coincide with A. § 28, 2

(Two straight lines determine a point.)

5. $\therefore \triangle ABC \cong \triangle A'B'C'$, by definition of congruence. § 57

Note. Prop. II, and prop. III following, are attributed to Thales.

Exercises. 15. In the figure on p. 19, given that OA bisects angle XOY, and that OB is perpendicular to OA, prove that OB bisects angle YOX'.



16. Show that the distance BA across a lake may be measured by setting up a stake at O, sighting across it to fix the lines $A'B$ and $B'A$, laying off $OA' = OA$, and $OB' = OB$, and then measuring $B'A'$.

64. Reciprocal Theorems. The student will notice that propositions I and II have a certain similarity. Indeed, if the words *side* and *angle* are interchanged in prop. I, it becomes prop. II, and if interchanged in prop. II that becomes prop. I. Theorems of this kind are called **reciprocal.** The relation is more clearly seen by resorting to parallel columns.

Prop. I. If two triangles have two *sides* and the included *angle* of the one respectively equal to two *sides* and the included *angle* of the other, the triangles are congruent.

Prop. II. If two triangles have two *angles* and the included *side* of the one respectively equal to two *angles* and the included *side* of the other, the triangles are congruent.

Moreover, if small letters and capitals are interchanged in the proof of prop. I, the proof becomes that of prop. II.

65. The principle involved is called the **Principle of Reciprocity,** and is extensively used in geometry. But the student must not suppose that because a theorem is true its reciprocal theorem is also true; in elementary geometry, involving measurements, the reciprocal is often false. The principle is, however, of great value even here, for it leads the student to see the relation between propositions, and *it often suggests new possible theorems for investigation.* For these purposes we shall use it.

At present it is sufficient to say that for many theorems of plane geometry reciprocal theorems may be formed by replacing the words

point by *line*,

line by *point*,

angles of a triangle by (*opposite*) *sides of a triangle*.

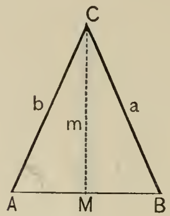
sides of a triangle by (*opposite*) *angles of a triangle*.

Exercises. 17. Explain this statement and tell why it is true: Any two sides and the included angle of a triangle determine the remaining parts.

18. State the reciprocal of ex. 17 and tell whether it is true, and why.

Proposition III.

66. Theorem. *If two sides of a triangle are equal, the angles opposite those sides are equal.*



Given the $\triangle ABC$ with $AC = BC$.

To prove that $\angle A = \angle B$.

Proof. 1. Suppose m to bisect $\angle ba$.

2. Then $\because b = a$, Given

and $\angle bm = \angle ma$,

and $m \equiv m$,

3. $\therefore \triangle AMC \cong \triangle BMC$, Prop. I

(State prop. I.)

and $\angle A = \angle B$, by definition of congruence. § 57

Corollary. *If a triangle is equilateral, it is also equiangular.*

For by the theorem the angles opposite the equal sides are equal.

67. Definitions. The line from any vertex of a triangle to the mid-point of the opposite side is called the **median** to that side.

In the above figure, CM is the median to AB.

If a triangle has two equal sides, it is called an **isosceles triangle.**

The third side is called the *base of the isosceles triangle*, and the equal sides are called the *sides*.

A triangle which has no two sides equal is called a **scalene triangle.**

The **distance** from one point to another is the length of the straight line-segment joining them.

The **distance** from a point to a line is the length of the perpendicular from that point to that line.

That this perpendicular is unique will be proved later.

This is the meaning of the word *distance* in plane geometry. In speaking of points on a curved surface (for example, the earth's surface), distance may be measured on a curved line.

68. In the figure of prop. III,

$$\triangle AMC \cong \triangle BMC, \text{ as proved.}$$
$$\therefore AM = MB,$$
$$\text{and } \angle CMA = \angle BMC,$$

and hence each is a right angle.

In cases of this kind the points A and B are said to be **symmetric with respect to an axis.** Hence, in the figure, CM is called an **axis of symmetry.** And, in general, two *systems of points*, A_1, B_1, C_1, ..., A_2, B_2, C_2, ..., are said to be *symmetric with respect to an axis* when all lines, A_1A_2, B_1B_2, ..., are bisected at right angles by that axis.

Also, *two figures are said to be symmetric with respect to an axis* when their systems of points are symmetric.

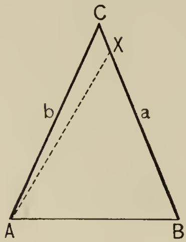
A *single figure*, like that of prop. III, is said to be *symmetric with respect to an axis* when this axis divides it into two symmetric figures.

Exercises. 19. If four lines go out from a point making four angles of which the first and third are equal, and the second and fourth are equal, prove that the four lines form two intersecting straight lines.

20. In the figure on p. 19, if a line passes through O and bisects angle XOA, prove that it also bisects angle $X'OC$.

Proposition IV.

69. Theorem. *If two angles of a triangle are equal, the sides opposite those angles are equal.*



Given the $\triangle ABC$ with $\angle A = \angle B$.

To prove that $a = b$.

Proof. 1. Suppose that $a \neq b$,

and that $a > b$.

2. Then let BX, a part of a, equal b, and join A and X

3. Then $\because \angle B = \angle BAC$, Given

and $AB \equiv AB$,

$\therefore \triangle ABC \cong \triangle BAX$. Why?

4. $\therefore$ the supposition leads to an absurdity, for

$\triangle ABC > \triangle BAX$, Ax. 8

(State ax. 8.)

and $\therefore a \ngtr b$.

In the same way it may be shown that $a \nless b$,

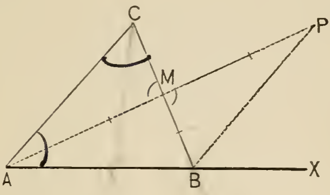
and $\therefore a = b$.

Corollary. *If a triangle is equiangular, it is also equilateral.* (Why?)

Exercise. 21. If four points, A, B, C, D, are placed in order on a line, and if $AC = BD$, prove that $AB = CD$.

Proposition V.

70. Theorem. *If any side of a triangle is produced, the exterior angle is greater than either of the interior angles not adjacent to it.*



Given the $\triangle ABC$, with AB produced to X.

To prove that $\angle XBC > \angle C$, and also $> \angle BAC$.

Proof. 1. Suppose BC bisected at M. AM drawn and produced to P so that $MP = AM$, and BP drawn.

2. Then $\because \angle BMP = \angle CMA$, Why?

$\therefore \triangle BPM \cong \triangle CAM$, Why?

and $\angle PBM = \angle C$. § 57

3. But $\angle XBC > \angle PBM$. Why?

4. $\therefore \angle XBC > \angle C$. Why?

5. Similarly, by producing CB, bisecting AB at N, producing CN, etc., it can be shown that an angle equal to $\angle XBC$ is greater than $\angle BAC$.

Exercises. 22. Show that, in the figure of prop. V, $\angle XBC > \angle BAC$ by following out in full the proof suggested in step 5.

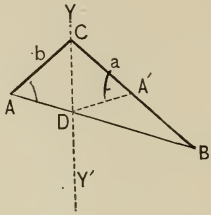
23. In the figure of prop. V, join C to any point in the segment AB and prove that $\angle CBA + \angle BAC < 180°$.

24. If a diagonal of a quadrilateral bisects two angles, the quadrilateral has two pairs of equal sides.

25. How many equal lines can be drawn from a given point to a given line? Show that if another is supposed to be drawn, an absurdity results.

PROPOSITION VI.

71. Theorem. *If two sides of a triangle are unequal, the opposite angles are unequal and the greater side has the greater angle opposite.*



Given the $\triangle ABC$, with $a > b$.

To prove that $\angle A > \angle B$.

Proof. 1. Suppose $\angle C$ bisected by YY' cutting AB at D, CA' made equal to CA, and DA' drawn.

2. Then $\triangle ADC \cong \triangle A'DC$, and $\angle A = \angle CA'D$. Why?

3. But $\angle CA'D > \angle B$. Prop. V

(§ 70. If any side of a $\triangle$ is produced, the exterior angle is greater than either of the int. $\angle$s not adjacent to it.)

4. $\therefore \angle A > \angle B$. Subst. 2 in 3

Exercises. 26. State, without proof, the reciprocal of prop. VI.

27. Can a scalene triangle have two equal angles? Proof.

28. Prove prop. VI by drawing AA' instead of DA', and proving that $\angle A > \angle A'AC = \angle CA'A > \angle B$.

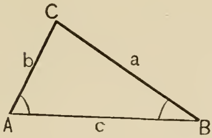
29. $ABCD$ is a quadrilateral of which DA is the longest side and BC the shortest. Which is greater, $\angle B$ or $\angle D$? Prove it. (Suggestion: Draw BD.) Also $\angle C$ or $\angle A$? Prove it.

30. How many perpendiculars can be drawn to a given line from a point outside that line? Show that any other supposition violates prop. V.

31. ABC is a triangle having $\angle B =$ twice $\angle A$; $\angle B$ is bisected by a line meeting b at D; prove that $AD = BD$.

PROPOSITION VII.

72. Theorem. *If two angles of a triangle are unequal, the opposite sides are unequal and the greater angle has the greater side opposite.*



Given the $\triangle ABC$ with $\angle A > \angle B$.

To prove that $a > b$.

Proof. 1. $a \neq b$, for if $a = b$, then $\angle A = \angle B$. Why?

2. $a \nless b$, for if $a < b$, then $\angle A < \angle B$. Prop. VI. State it.

3. $\therefore a$ must be greater than b.

NOTE. It must not be inferred from props. VI, VII that, because one angle of a triangle is twice as large as another, one side is twice as long as another.

Exercises. 32. Prove that if the bisector of any angle of a triangle is perpendicular to the opposite side, the triangle is isosceles.

33. Suppose any point taken on the perpendicular bisector of a line; is it equally or unequally distant from the ends of the line? Give the proof in full.

34 *a.* Prove that in an isosceles triangle ABC, where $a = b$, the bisector of $\angle C$, produced to c, bisects *side* c.

34 *b.* Prove that in an isosceles triangle abc, where $\angle A = \angle B$, the bisector of *side* c, joined to C, bisects $\angle C$.

35. After reading § 73, state the converse of each of the following: (*a*) prop. III; (*b*) prop. IV; (*c*) prop. VI; (*d*) prop. VII; (*e*) this statement, If the animal is a horse, then the animal has two eyes. Of these converses, how many are true?

36. What kind of a triangle is formed by joining the mid-points of the sides of an equilateral triangle? Prove it.

73. The Law of Converse. Two theorems are said to be the **converse,** each of the other, when what is given in the one is what is to be proved in the other, and *vice versa.*

E.g. props. VI and VII. The converse of a theorem must not be confused with its reciprocal. Props. I and II are reciprocal, but not converse.

Because a theorem is true its converse is not necessarily true.

For example, prel. prop. I may be stated thus: Given that $\angle s$ r and r' are rt. $\angle s$, to prove that $\angle r = \angle r'$; the converse is, Given that $\angle r = \angle r'$, to prove that they are rt. $\angle s$. This converse is evidently false, for $\angle r$ could equal $\angle r'$ without their being rt. $\angle s$.

But there is one important class of converse theorems, illustrated by props. IV and VII, that should be mentioned. *Whenever three theorems have the following relations, their converses must be true:*

1. If it has been proved that when $A > B$, then $X > Y$, and
2. " " " $A = B$, " $X = Y$, "
3. " " " $A < B$, " $X < Y$,

then the converse of each of these is true. For

1′. If $X > Y$, then A can neither be equal to nor less than B, without violating 2 or 3; $\therefore A > B$. (Converse of 1.)

2′. If $X = Y$, then A can neither be greater nor less than B, without violating 1 or 3; $\therefore A = B$. (Converse of 2.)

3′. If $X < Y$, then A can neither be greater than nor equal to B, without violating 1 or 2; $\therefore A < B$. (Converse of 3.)

The law just proved will hereafter be referred to as the **Law of Converse.** By its use the proof of the converse of many theorems, where true, is made very simple.

The student should not proceed further unless the Law of Converse is thoroughly understood, and its proof mastered.

Prop. VII may now be proved by the Law of Converse, thus:

If $a > b$, then $\angle A > \angle B$.	Prop. VI
If $a = b$, " $\angle A = \angle B$.	" III
If $a < b$, " $\angle A < \angle B$.	" VI

$\therefore$ each converse is true, and if $\angle A > \angle B$, then $a > b$.

74. Suggestions as to the Treatment of the Exercises. Thus far the student has been left to his own ingenuity in treating the exercises. A few suggestions should now be given.

1. *In attacking a theorem take the most general figure possible.*

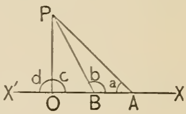
E.g. if a theorem relates to a triangle, draw a *scalene* triangle; an equilateral or an isosceles triangle often deceives the eye, and leads away from the demonstration. *Draw all figures accurately;* an accurate figure often *suggests* the demonstration. But the student who relies too much upon the accuracy of the figure in the demonstration itself is liable to go astray.

2. *Be certain that what is given and what is to be proved are clearly stated, with reference to the letters of the figure.*

This has been done in all of the theorems thus far proved. The neglect to do so in the exercises is one of the most fruitful sources of failure.

3. *Then begin by assuming the theorem true; see what follows from that assumption; then see if this can be proved true without the assumption; if so, try to reverse the process.*

E.g. suppose $PO \perp X'X$, and PB, PA two obliques cutting off $OA > OB$, as in the figure, and that it is required to prove $PA > PB$. Assume it true; then $\angle b > \angle a$. Now see if $\angle b > \angle a$ *without* the assumption; $\angle b > \angle c$, which $= \angle d$, which $> \angle a$, by prop. V; $\therefore \angle b > \angle a$, without the assumption. Now reverse the process; $\because \angle b > \angle a$, $\therefore PA > PB$ by prop. VII.



4. *Or begin by assuming the theorem false, and endeavor to show the absurdity of the assumption.* (*Reductio ad absurdum.*)

5. To secure a clearer understanding of the theorem it is often well to follow Pascal's advice and *substitute the definition for the name of the thing defined.*

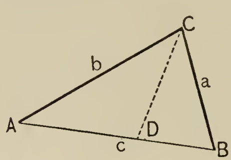
E.g. suppose it is to be proved that the median to the base of an isosceles triangle is perpendicular to the base. Instead of saying:

"Given CM the median to the base of the isosceles triangle ABC" (see figure on p. 28), it is often better to say:

"Given $\triangle ABC$, with $AC = BC$, and M taken on AB so that $AM = MB$," for then the facts stand out prominently without any confusing terms.

Proposition VIII.

75. Theorem. *The sum of any two sides of a triangle is greater than the third side.*



Given the $\triangle ABC$.

To prove that $a + b > c$.

Proof. 1. Suppose $\angle C$ bisected by CD.

Then $\angle CDA > \angle DCB$. Prop. V. State it

2. And $\because \angle ACD = \angle DCB$, Step 1

$\therefore \angle CDA > \angle ACD$.

$\therefore b > AD$. Prop. VII. State it

Similarly, $a > DB$.

3. $\therefore a + b > c$.

Corollary. *The difference of any two sides of a triangle is less than the third side.*

For if $a + b > c$, and $c > b$, then $a > c - b$, by ax. 5.

Exercises. 37. Two equal lines, AC and AD, are drawn on opposite sides of a line AB and making equal angles with it; BC and BD are drawn. Show that BC and BD also make equal angles with AB.

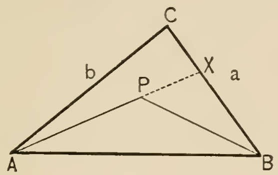
38. P, Q, R are points on the sides AB, BC, CA, respectively, of an equilateral triangle ABC, such that $AP = BQ = CR$; joining P, Q, and R, prove that $\triangle PQR$ is equilateral. (Notice that ex. 36 is merely a special case of this one.)

39 *a*. The bisectors of the equal *angles* of an isosceles triangle form, with the *base*, an isosceles triangle.

39 *b*. The mid-points of the equal *sides* of an isosceles triangle form, with the *vertex*, the vertices of an isosceles triangle.

PROPOSITION IX.

76. Theorem. *If from the ends of a side of a triangle two lines are drawn to a point within the triangle, their sum is less than the sum of the other two sides of the triangle, but they contain a greater angle.*



Given the $\triangle ABC$, P a point within, and BP and PA drawn.

To prove that (1) $BP + PA < a + b$, (2) $\angle APB > \angle C$.

Proof. 1. Produce AP to meet a at X.

Then

$XP + PA = XA < XC + b$, Ax. 8; prop. VIII

(State ax. 8 and prop. VIII.)

and $BP < BX + XP$. Prop. VIII

2. $\therefore BP + XP + PA < BX + XC + XP + b$.

3. $\therefore BP + PA < a + b$, which proves (1). Why?

4. Also,

$\angle APB > \angle PXB > \angle C$, which proves (2). Why?

Exercises. 40 *a.* If the equal *sides* of an isosceles triangle are bisected, the *lines* joining the *points* of bisection with the vertices of the equal *angles* are equal.

40 *b.* If the equal *angles* of an isosceles triangle are bisected, the *angles* formed by the *lines* of bisection and the equal *sides* are equal.

41. The perimeter of a quadrilateral is less than twice the sum of its two diagonals.

Proposition X.

77. Theorem. *If two triangles have two sides of the one respectively equal to two sides of the other, but the included angles unequal, then the third sides are unequal, the greater side being opposite the greater angle.*

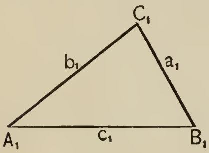


Fig. 1.

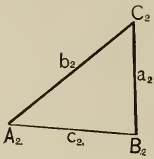


Fig. 2.

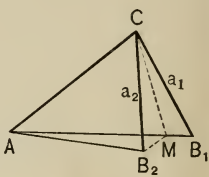


Fig. 3.

Given the ⧍ $A_1B_1C_1$ and $A_2B_2C_2$, with $a_1 = a_2$, $b_1 = b_2$, but $\angle C_1 > \angle C_2$.

To prove that $c_1 > c_2$.

Proof. 1. Suppose $\triangle A_2B_2C_2$ placed on $\triangle A_1B_1C_1$ so that b_2 and b_1, being equal, coincide. § 61

Then $\because \angle C_1 > \angle C_2$, side a_2 must fall within $\angle C_1$, as in Fig. 3.

2. Suppose CM drawn bisecting $\angle B_2CB_1$, and B_2M drawn.

3. Then in ⧍ B_1MC, B_2MC,

$CB_1 = CB_2$, Given

$CM \equiv CM$,

$\angle MCB_1 = \angle B_2CM$. Step 2

4. $\therefore \triangle B_1MC \cong \triangle B_2MC$,

and $MB_1 = MB_2$. Prop. I

5. But $AM + MB_2 > AB_2$. Prop. VIII

$\therefore AM + MB_1 > AB_2$,

or $c_1 > c_2$.

The proof is the same when B_2 falls above A_1B_1.

PROPOSITION XI.

78. Theorem. *If two triangles have two sides of the one respectively equal to two sides of the other, but the third sides unequal, then the included angles are unequal, the greater angle being opposite the greater third side.*

Given $\triangle$ $A_1B_1C_1$ and $A_2B_2C_2$, with $a_1 = a_2$, $b_1 = b_2$, $c_1 > c_2$.

To prove that $\angle C_1 > \angle C_2$.

Proof. 1. It has been shown that if $a_1 = a_2$, $b_1 = b_2$, and if $\angle C_1 > \angle C_2$, then $c_1 > c_2$. Prop. X

2. And if $\angle C_1 =$ " " " $=$ " Prop. I

3. " " $\angle C_1 <$ " " " $<$ " Prop. X

4. $\therefore$ the converses are true, which proves the theorem. § 73. Law of Converse

(Explain the Law of Converse. Since this law is so often used, it should be reviewed frequently.)

Exercises. 42. Are props. X and XI reciprocals? converses?

43. In $\triangle ABC$, suppose $CA > AB$, and that points P, Q are taken on AB, CA respectively, so that $PB = CQ$. Prove that $BQ < CP$.

44. Investigate ex. 43 when P is taken on AB produced, and Q on AC produced.

45. The equal sides, AC, BC, of an isosceles triangle ABC are produced through the vertex to P and Q respectively, so that $AP = BQ$. Prove that $BP = AQ$.

46. Prove that the straight line joining any two points is less than any broken line joining them.

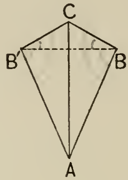
47. Prove that the perimeter of a triangle is less than twice the sum of the three medians.

48. In a quadrilateral, prove that the sum of either pair of opposite sides is less than the sum of its two diagonals.

49. If the perpendicular from any vertex of a triangle to the opposite side divides that side into two segments, how does each of these segments compare in length with its adjacent side of the triangle? Prove it.

Proposition XII.

79. Theorem. *If two triangles have the three sides of the one respectively equal to the three sides of the other, the triangles are congruent.*



Given $\triangle ABC$, $AB'C$, with $AB = AB'$, $BC = B'C$, and $AC = AC$.

To prove that $\triangle ABC \cong \triangle AB'C$.

Proof. 1. Suppose no side longer than AC. Then the $\triangle$, being mutually equilateral, may be placed with AC in common, and on opposite sides of AC. Draw BB'.

2. Then $\angle CBB' = \angle BB'C$, Prop. III
and $\angle B'BA = \angle AB'B$. Why?

3. $\therefore \angle CBA = \angle AB'C$. Why?

4. $\therefore \triangle ABC \cong \triangle AB'C$. Why?

AC is evidently an axis of symmetry (§ 68) in the figure.

Exercises. 50. Suppose three sticks to be hinged together to form a triangle, could the sides be moved so as to change the angles? On what theorem does the answer depend? How would it be with a hinged quadrilateral?

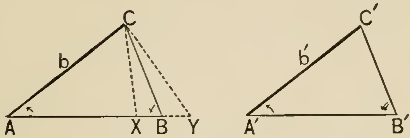
51. Ascertain and prove whether or not a quadrilateral is determined when the four sides and either diagonal are given in fixed order.

52. Also when the four sides and one angle are given in fixed order.

53. How many braces would it take to stiffen a three-sided plane figure? four-sided? five-sided?

PROPOSITION XIII.

80. Theorem. *If two triangles have two angles of the one respectively equal to two angles of the other, and the sides opposite one pair of equal angles equal, the triangles are congruent.*



Given $\triangle ABC, A'B'C'$, with $\angle A = \angle A'$, $\angle B = \angle B'$, $b = b'$.

To prove that $\triangle ABC \cong \triangle A'B'C'$.

Proof. 1. Place $\triangle A'B'C'$ on $\triangle ABC$ so that A' falls at A, $A'B'$ lies along AB, and C and C' both lie on the same side of AB. § 61

2. Then because $\angle A = \angle A'$, and $b = b'$, b' coincides with b, and C' with C.

3. Now B' cannot fall between A and B, as at X, for then $\angle CXA$, which $= \angle B'$, would be greater than $\angle B$. Prop. V. State it

4. Neither can B' fall on AB produced, as at Y, for then $\angle Y$, which $= \angle B'$, would be less than $\angle B$. Prop. V

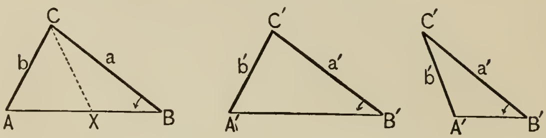
5. $\therefore B'$ must fall at B, and the $\triangle$ are congruent. § 57

Exercises. 54. If YO meets $X'X$ at O, and YA, YB are drawn meeting $X'X$ at A, B; and if $YA = YB$, and $AO \neq OB$, which is the greater, $\angle AYO$ or $\angle OYB$?

55. Consider the diagonals of an equilateral quadrilateral, (a) as to their bisecting each other, (b) as to the kind of angles they make with each other. State the theorems which you discover and prove them.

Proposition XIV.

81. Theorem. *If two triangles have two sides of the one respectively equal to two sides of the other, and the angles opposite one pair of equal sides equal, then the angles opposite the other pair of equal sides are either equal or supplemental, and if equal the triangles are congruent.*



Given $\triangle\!\!\triangle$ ABC, $A'B'C'$, with $a = a'$, $b = b'$, $\angle B = \angle B'$.

To prove that either (1) $\angle A = \angle A'$ and $\triangle ABC \cong \triangle A'B'C'$,

or (2) $\angle A + \angle A' =$ st. $\angle$.

Proof. 1. Place $\triangle A'B'C'$ on $\triangle ABC$ so that B' falls at B, a' coincides with its equal a, and A' and A fall on the same side of a. § 61

2. Then $\because \angle B = \angle B'$, $B'A'$ lies along BA.

3. Then either A' falls at A, the $\triangle\!\!\triangle$ are congruent and $\angle A = \angle A'$; or else A' falls at some other point on BA, as at X, and $\triangle A'B'C' \cong \triangle XBC$.

4. But $\because CX = b' = b$,

$\therefore \angle A = \angle CXA$. Prop. III. State it

5. And

$\because \angle CXA + \angle BXC =$ st. $\angle$, § 14, def. st. $\angle$

$\therefore \angle A + \angle A' =$ st. $\angle$.

Exercises. 56. In prop. XIV, step 3, X may lie on BA produced, in some cases. Draw the figure and prove.

57. In prop. XIV prove that $\angle A$ and $\angle A'$ must be equal, if (1) they are of the same species (*i.e.* both right, both acute, or both obtuse); or (2) angles B and B' are both right angles; or (3) $b \nless a$.

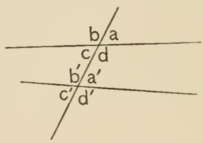
2. PARALLELS AND PARALLELOGRAMS.

82. Definitions. If two straight lines in the same plane do not meet, however far produced, they are said to be **parallel**.

E.g. A and B in the annexed figure. The fact that A is parallel to B is indicated by the symbol $A \parallel B$.

A line cutting two or more lines is called a **transversal** of those lines.

In the figure of the parallel lines, T is a transversal of A and B. The adjacent figure shows a transversal of two non-parallel lines. The figure on p. 46 shows a transversal of three lines.



The angles formed by a transversal cutting two lines (parallel or not) have received special names. Thus, in the annexed figure,

a, b, c', d' are called **exterior** angles;

a', b', c, d are called **interior** angles;

a and c' are called **alternate** angles; also b and d', c and a', b' and d;

a and a' are called **corresponding** angles; also b and b', c and c', d and d'.

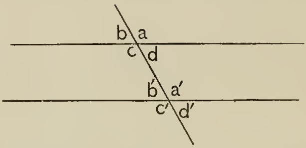
Exercises. 58. Angle A, of triangle ABC, is bisected by a line meeting BC at P. Which is the longer, AB or BP? Prove it. Also CA or PC? Prove it.

59. State the reciprocal of prop. XIII, and tell whether it is true without modification. In what proposition is your statement proved?

60. If a quadrilateral has two pairs of equal sides, prove that it must have one pair and may have two pairs of equal angles, depending upon the arrangement of the sides.

Proposition XV.

83. Theorem. *If a transversal of two lines makes a pair of alternate angles equal, then (1) any angle is equal to its alternate angle, (2) any angle is equal to its corresponding angle, and (3) any two interior, or any two exterior, angles on the same side of the transversal are supplemental.*



Given a transversal cutting two lines, making equal alternate $\angle s$ d and b' as in the figure.

To prove that (1) $\angle a = \angle c'$,

(2) $\angle a = \angle a'$,

(3) $\angle b + \angle c' =$ st. $\angle$.

Proof. 1. $\angle d + \angle a =$ st. $\angle$,

and $\angle b' + \angle c' =$ st. $\angle$. § 14, def. st. $\angle$

2. $\therefore \angle d + \angle a = \angle b' + \angle c'$. Why?

3. $\therefore \angle a = \angle c'$, which proves (1). Ax. (?)

4. $\because \angle c' = \angle a'$, $\therefore \angle a = \angle a'$, which proves (2).

5. Now $\because \angle b' = \angle d$, and $\angle d = \angle b$, Why?

$\therefore \angle b' = \angle b$. Why?

But $\because \angle b' + \angle c' =$ st. $\angle$, § 14, def. st. $\angle$

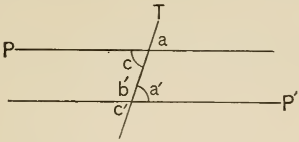
$\therefore \angle b + \angle c' =$ st. $\angle$.

Corollaries. 1. *If two corresponding angles are equal, the same three conclusions follow.*

2. *If two interior or two exterior angles on the same side of the transversal are supplemental, the same conclusions follow.*

PROPOSITION XVI.

84. Theorem. *If a transversal of two lines makes a pair of alternate angles equal, the two lines are parallel.*



Given P and P', two lines, cut by a transversal T, making equal alternate $\angle$s c and a'.

To prove that $P \parallel P'$.

Proof. 1. P, P' cannot meet towards P', for then $\angle c$ would be an ext. $\angle$ of a $\triangle$, and $\therefore \angle c$ would be greater than $\angle a'$. Prop. V. State it

2. P, P' cannot meet towards P, for then $\angle a'$ would be greater than $\angle c$. Why?

3. $\therefore P$, P' cannot meet at all, and $P \parallel P'$. Def. parallel

Similarly any other two alt. $\angle$s may be taken equal.

COROLLARIES. 1. *If two corresponding angles are equal, the lines are parallel.*

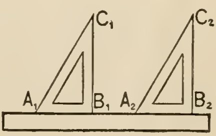
For then two alt. $\angle$s are equal. Prop. XV, cor. 1, which says — (?)

2. *If two interior or two exterior angles on the same side of the transversal are supplemental, the lines are parallel.*

For then two alt. $\angle$s are equal. Prop. XV, cor. 2, which says — (?)

3. *Two lines perpendicular to the same line are parallel.* (Why?)

Exercises. 61. In prop. XVI would lines bisecting $\angle a'$ and $\angle c$ be parallel? Prove it.



62. Show that if a draughtsman's square slides along a ruler, as in the annexed figure, $B_1C_1 \parallel B_2C_2$, and $A_1C_1 \parallel A_2C_2$.

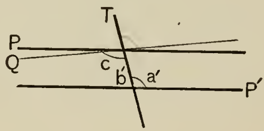
85. Postulate of Parallels. It now becomes necessary to assume another postulate, and upon it rests much of the elementary theory of parallels. It is: *Two intersecting straight lines cannot both be parallel to the same straight line.*

COROLLARY. *A line cutting one of two parallel lines cuts the other also, the lines being unlimited.*

(Show that the corollary is necessarily true if the postulate is.)

PROPOSITION XVII.

86. Theorem. *The alternate angles formed by a transversal with two parallels are equal.*



Given P and P' two parallels, and T, a transversal.

To prove that any $\angle c$ equals its alternate $\angle a'$.

Proof. 1. Suppose $\angle c > \angle a'$, and that Q is drawn as in the figure, making an $\angle$ equal to $\angle a'$.

2. Then Q would be parallel to P'. Why?

3. But this would be impossible, $\because P \parallel P'$. § 85

(Two intersecting straight lines cannot both be parallel to the same straight line.)

4. Similarly, it is absurd to suppose that $\angle a' > \angle c$.

$$\therefore \angle c = \angle a'.$$

COROLLARIES. 1. *A line perpendicular to one of two parallels is perpendicular to the other also.*

For it cuts the other (§ 85, cor.) and the alternate angles are equal right angles.

2. *A line cutting two parallels makes corresponding angles equal, and the interior, or the exterior, angles on the same side of the transversal supplemental.*

For the alternate angles are equal (prop. XVII), and hence prop. XV applies.

3. *If the alternate or the corresponding angles are unequal, or if the interior angles on the same side of the transversal are not supplemental, then the lines are not parallel, but meet on that side of the transversal on which the sum of the interior angles is less than a straight angle.*

For the lines cannot be parallel, by prop. XVII and cor. 2.

Further, suppose $\angle c + \angle b' < \text{st.} \angle$;

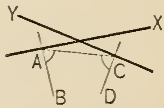
then $\because \angle a' + \angle b' = \text{st.} \angle$,

it follows that $\angle c < \angle a'$.

$\therefore$ P and P' cannot meet towards P', for then $\angle c$ would be greater than $\angle a'$, prop. V.

Let the student give the proof in full form, in steps.

4. *Two lines respectively perpendicular to two intersecting lines cannot be parallel.*



For, in the annexed figure, let $AB \perp X$, $CD \perp Y$; join A and C. Then $\angle BAC <$ rt. $\angle$, and $\angle ACD <$ rt. $\angle$; $\therefore$ their sum is $<$ st. $\angle$; $\therefore$ cor. 3 applies.

Give proof in full form in steps.

5. *If the arms of one angle are parallel or perpendicular to the arms of another, the angles are equal or supplemental.*

The proof is left to the student.

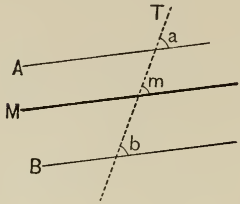
Exercises. 63. In the figure of prop. XV, suppose $a = c' = 120° \, 30'$, how large is each of the other angles?

64. In the same figure, suppose $a + d' =$ st. $\angle$, and $a = 2\,d$, how large is each of the other angles?

65. If a transversal cuts two lines making the sum of the two interior angles on the same side of the transversal a straight angle, one of them being $30° \, 27'$, how large is each of the other angles?

Proposition XVIII.

87. Theorem. *Lines parallel to the same line are parallel to each other.*



Given $A \parallel M$, and $B \parallel M$.

To prove that $A \parallel B$.

Proof. 1. Suppose T a transversal, making corresponding $\angle$s a, m, b, with A, M, B, respectively.

2. Then $\because A \parallel M$,

$\therefore \angle a = \angle m$. Prop. XVII, cor. 2

3. And $\because B \parallel M$, $\therefore \angle b = \angle m$.

4. $\therefore \angle a = \angle b$. Why?

5. $\therefore A \parallel B$. Prop. XVI, cor. 1. State it

Exercises. 66. In prop. XVIII, if T cuts A, must it necessarily cut M? Why? If it cuts M, must it necessarily cut B? Why?

67. Prove that a line parallel to the base of an isosceles triangle makes equal angles with the sides or the sides produced. (The line may pass above, through, or below the triangle, or through the vertex.)

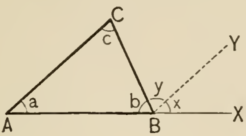
68. If through any point equidistant from two parallels, two transversals are drawn, prove that they will cut off equal segments of the parallels.

69. ABC is a triangle, and through P, the point of intersection of the bisectors of $\angle B$ and $\angle C$, a line is drawn parallel to BC, meeting AB at M, and CA at N. Prove that $MN = MB + CN$.

70. Through the mid-point of the segment of a transversal cut off by two parallels, a straight line passes, terminated by the parallels. Prove that this line is bisected by the transversal.

PROPOSITION XIX.

88. Theorem. *In any triangle, (1) any exterior angle equals the sum of the two interior non-adjacent angles; (2) the sum of the three interior angles is a straight angle.*



Given $\triangle ABC$, with AB produced to X.

To prove that (1) $\angle XBC = \angle A + \angle C$;

(2) $\angle A + \angle B + \angle C =$ st. $\angle$.

Proof. 1. Suppose $BY \parallel AC$, and $\measuredangle$ named as in the figure.

2. Then $\angle x = \angle a$, Why?

and $\angle y = \angle c$. Why?

3. $\therefore \angle x + \angle y$, or $\angle XBC$, $= \angle a + \angle c$, which proves (1). Ax. 2

4. But $\angle x + \angle y + \angle b =$ st. $\angle$. Def. st. $\angle$

5. $\therefore \angle a + \angle b + \angle c =$ st. $\angle$, by substituting 3 in 4, which proves (2).

NOTES. 1. Prop. XIX, (2), is attributed to Pythagoras.

2. The theorem is one of the most important of geometry. To it and to its corollaries (p. 50) frequent reference is hereafter made.

Exercises. 71. *PQR* is a triangle having $PQ = PR$; RP is produced to S so that $PS = RP$; QS is drawn. Prove that $QS \perp RQ$.

72. Prove prop. XIX, (2), by drawing through C, in the figure given, a line $\parallel AB$.

73. Also by assuming any point P on AB, drawing PC, and showing that $\angle BPC + \angle CPA =$ st. $\angle$, and also equals the sum of the interior angles.

74. State the reciprocal of prop. VIII, and prove or disprove it.

Corollaries to prop. XIX. 1. *If a triangle has one right angle, or one obtuse angle, the other angles are acute.*

For the sum of all three is a straight angle.

2. *Every triangle has at least two acute angles.*

For if it had none or only one, the sum of the others would equal or exceed what kind of an angle, and thus violate what theorem?

3. *From a point outside a given line not more than one perpendicular can be drawn to that line.*

For if two could be drawn, a triangle could be formed having how many right angles, thus violating what corollary?

4. *If a triangle has a right angle, the two acute angles are complemental.*

For the sum of all three must equal two right angles; therefore, etc.

5. *If two triangles have two sides of the one respectively equal to two sides of the other, and the angles opposite one pair of equal sides right angles, or equal obtuse angles, the triangles are congruent.*

For prop. XIV then applies; the oblique angles cannot be supplemental.

6. *If two angles of one triangle equal two angles of another, the third angles are equal.* (Why?)

7. *Two triangles are congruent if two angles and any side of the one are respectively equal to the corresponding parts of the other.* (Why?)

8. *Each angle of an equilateral triangle is one-third of a straight angle.* (Why?)

89. Definitions. A triangle, one of whose angles is a right angle, is called a **right-angled triangle.**

A triangle, one of whose angles is an obtuse angle, is called an **obtuse-angled triangle.**

A triangle, all of whose angles are acute, is called an **acute-angled triangle.**

The side opposite the right angle of a right-angled triangle is called the **hypotenuse.**

90. SUMMARY OF PROPOSITIONS CONCERNING CONGRUENT TRIANGLES. Two triangles are congruent if the following parts of the one are equal to the corresponding parts of the other:

1. *Two sides and the included angle.* Prop. I
2. *Two angles and the included side.* Prop. II
3. *Three sides.* Prop. XII
4. *Two angles and the side opposite one,* Prop. XIII
 or, more generally, *two angles and a side.* Prop. XIX, cor. 7
5. *Two sides and the angle opposite one, provided that angle is not acute.* Prop. XIV, and prop. XIX, cor. 5

If the angle is acute, then from two sides and the acute angle opposite one of them two different triangles may be possible. This is therefore known as the *ambiguous case.* If the side opposite the acute angle is not less than the given adjacent side, the case is not ambiguous. Why? Draw the figures illustrating the ambiguous case.

These propositions can be summarized in one general proposition: *A triangle is determined when any three independent parts are given, except in the ambiguous case.*

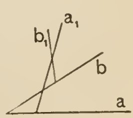
It should be noted that the three angles are not three independent parts, since when any two of them are given the third is determined. (Prop. XIX.)

Exercises. 75. In a right-angled triangle, the mid-point of the hypotenuse is equidistant from the three vertices. (Suppose a line drawn from the vertex C of the right angle making with a an angle equal to $\angle B$.)

76. In a right-angled triangle, a perpendicular let fall from the vertex of the right angle, upon the hypotenuse, cuts off two triangles mutually equiangular to the original triangle.

77. If $a \perp x$ and $b \perp y$, and x intersects y, then $\angle ab = \angle xy$.

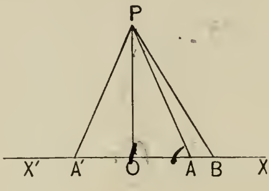
78. In the annexed figure, $\angle aa_1 = \angle bb_1$. Prove that (1) $\angle aa_1 = \angle ab + \angle ba_1$; (2) $\angle bb_1 = \angle ba_1 + \angle a_1b_1$.



79. How many degrees in each angle of an isosceles right-angled triangle? also of an isosceles triangle whose vertical angle is 72°? 178°? 60°?

Proposition XX.

91. Theorem. *Of all lines drawn to a given line from a given external point, the perpendicular is the shortest; of others, those making equal angles with the perpendicular are equal; and of two others, that which makes the greater angle with the perpendicular is the greater.*



Given $PO \perp XX'$; PA, PA', PB, oblique to XX', with $\angle A'PO = \angle OPA < \angle OPB$.

To prove that (1) $PO < PA$,

(2) $PA' = PA$,

(3) $PB > PA$, or PA'.

Proof. 1. $\angle PAO < \angle AOP$. Prop. XIX, cor. 1

2. $\therefore PO < PA$, which proves (1). Prop. VII

3. $\angle AOP = \angle POA'$, Prel. prop. I

$\angle A'PO = \angle OPA$, Why?

and $PO \equiv PO$.

4. $\therefore \triangle AOP \cong \triangle A'OP$,

and $PA' = PA$, which proves (2). Why?

5. $\angle BAP$ is obtuse, $\because$ it $> \angle AOP$, Prop. V

and $\angle PBO$ is acute, $\because \angle BOP$ is rt. Why?

6. $\therefore PB > PA$, or its equal PA', by step 4, which proves (3). Prop. VII

COROLLARIES. 1. *From a given external point there can be two, and only two, equal obliques of given length to a given line.*

Prove it by a *reductio ad absurdum.*

2. *If from a point not on a perpendicular drawn to a line at its mid-point, lines are drawn to the ends of the line, these lines are unequal and the one cutting the perpendicular is the greater.*

Let Z be the point, not on OP in the figure. Suppose ZA' to cut OP at Y. Then $ZA' = ZY + YA > ZA$.

3. *The converse of cor. 2 is true.*

For if ZA' cuts OP, then $ZA' > ZA$, by cor. 2.
" ZA " " " " $<$ " " " "
" Z is on " " " $=$ " (Why ?)

∴ the Law of Converse (§ 73) evidently applies to this case.

4. *Of two obliques from a point to a line, that which meets the line at the greater distance from the foot of the perpendicular is the greater.*

For if $OB > OA$, then $\angle OPB > \angle OPA$. (Why ?) ∴ prop. XX applies.

5. *Two obliques from a point to a line, meeting that line at equal distances from the foot of the perpendicular, are equal, make equal angles with this line and also with the perpendicular.*

Give the proof in full.

6. *Two equal obliques from a point to a line cut off equal segments from the foot of the perpendicular.*

Draw the figure. It will then be seen that prop. XIX, cor. 5, applies. The ⊥ is evidently an axis of symmetry (§ 68).

Exercises. 80. A line perpendicular to the bisector of any angle of a triangle makes an angle with either arm of that angle equal to half the sum of the other two angles ; and, unless parallel to the base, it makes an angle with the line of the base equal to half the difference of those angles.

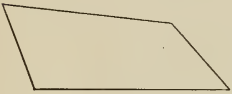
81. In an isosceles triangle, the perpendicular from the vertex, the median to the base, and the bisector of the vertical angle all coincide.

92. Definitions. A polygon is said to be **convex** when no side produced cuts the surface of the polygon.

A polygon is said to be **concave** when a side produced cuts the surface of the polygon.

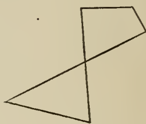
A polygon is said to be **cross** when the perimeter crosses itself.

The word *polygon* is understood, in elementary geometry, to refer to a convex or concave polygon unless the contrary is stated.



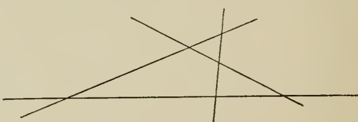
Convex.

Concave.



Cross.

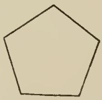
If all of the sides of a polygon are indefinitely produced, the figure is called a **general polygon.**



A general quadrilateral.

If a polygon is both equiangular and equilateral, it is said to be **regular.**

By the term *regular polygon*, a regular *convex* polygon is understood unless the contrary is stated.



Regular convex polygon.

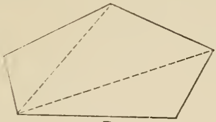
Regular cross polygon.

A polygon is called a *triangle*, *quadrilateral*, *pentagon*, *hexagon*, *heptagon*, *octagon*, *nonagon*, *decagon*, ... *dodecagon*, ... *pentedecagon*, ... *n-gon*, according as it has 3, 4, 5, 6, 7, 8, 9, 10, ... 12, ... 15, ... n-sides.

The student, even if unacquainted with Latin or Greek, should understand the derivation of these common terms. From the Latin are derived the words and prefix *tri-angle* (three-angle), *quadri-lateral* (four-side), *nona-* (nine); from the Greek are derived *poly-gon* (many-angle), *penta-* (five), *hexa-* (six), *hepta-* (seven), *octo-* (eight), *deca-* (ten), *dodeca-* (twelve), ... Much light will be thrown on the meaning of various geometric terms by consulting the Table of Etymologies in the Appendix.

PROPOSITION XXI.

93. Theorem. *The sum of the interior angles of an n-gon is (n – 2) straight angles.*



P

Given P, a polygon of n sides.

To prove that the sum of the interior angles is $(n-2)$ straight angles.

Proof. 1. P may be divided into $(n-2)$ ⊿s by diagonals which do not cross; for,

(*a*) A 4-gon (quadrilateral) is a △ + a △,
∴ 2 ⊿s, or $(4-2)$ ⊿s.

(*b*) A 5-gon (pentagon) is a 4-gon + a △,
∴ 3 ⊿s, or $(5-2)$ ⊿s.

(*c*) A 6-gon (hexagon) is a 5-gon + a △,
∴ 4 ⊿s, or $(6-2)$ ⊿s.

(*d*) And every addition of 1 side adds 1 △.

(*e*) ∴ for an n-gon there are $(n-2)$ ⊿s.

2. The sum of the ∠s of each △ is a st. ∠. Prop. XIX

3. ∴ the sum of the interior ∠s of an n-gon is $(n-2)$ st. ∠s, because these equal the sum of the ∠s of the ⊿s.

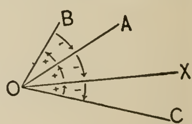
COROLLARY. *If each of two angles of a quadrilateral is a right angle, the other two angles are supplemental.* (Why?)

Exercise. 82. How many diagonals in a common convex pentagon? hexagon?

94. Generalization of Figures. If a thermometer registers 70° above zero, it is ordinarily stated, in scientific works, that it registers + 70°, while 10° below zero is indicated by − 10°, the sign changing from + to − as the temperature decreases through zero. Similarly, west longitude is represented by the sign +, while longitude on the other side of 0° (*i.e.* east) is represented by the sign −, the longitude changing its sign in passing through zero. So in speaking of temperature it is said that $10° + (-10°) = 0$, meaning thereby that if the temperature rises 10° from 0, and then falls 10°, the result of the two movements is the original temperature, 0.

This custom holds in geometry. Thus, in this figure, if the segment between B and C is thought of as extending from B to C, it would be named BC; and, as is usually done in geometry with lines thought of as extending to the right, it would be considered a *positive* line. But if it is thought of as extending from C to B, it would be named CB, and considered a *negative* line. Hence it is said that $BC + CB = 0$, an expression borrowed from algebra, where it would appear in a form like $x + (-x) = 0$.

Similarly, with regard to angles: the turning of an arm in a sense opposite to that of a clock-hand, counter-clockwise, is considered *positive*, while the turning in the opposite sense is considered *negative.* Thus, $\angle XOA$ is considered positive, but the acute $\angle AOX$ is considered negative, and this is indicated by the statement, $-\angle XOA =$ acute $\angle AOX$. Hence, as in the case of lines, $\angle XOA + (-\angle XOA) = \angle XOA +$ acute $\angle AOX =$ zero. On this account we pay special attention to the manner of lettering angles, distinguishing between $\angle XOA$ and $\angle AOX$. It is only recently that negative angles have been considered in elementary geometry, and hence the older works paid no attention to the order of the naming of the arms.



95. These considerations enable us to *generalize* many figures, with interesting results. Thus, prop. XXI is true for a cross polygon as well as for the simple cases usually considered. If, in Fig. 1, P is moved through AB to the position

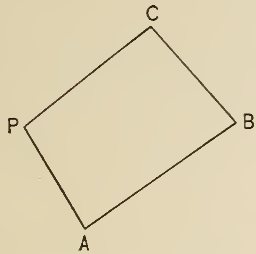


FIG. 1.

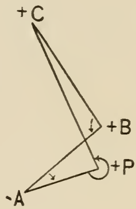


FIG. 2.

shown in Fig. 2, we shall still have $\angle A$ (which has passed through 0 and has become negative) $+ \angle B + \angle C + \angle P$ (which is now reflex) $= 2$ st. angles.

Exercises. 83. Prove the last statement made above.

84. How many points of intersection, at most, of the sides of a general quadrilateral? pentagon?

85. How many diagonals, at most, has a general quadrilateral?

86. Prove prop. XXI by connecting each vertex with a point O within the figure, thus forming n $\triangle$s, giving n st. $\angle$s, and then subtracting the two around O.

87. In an isosceles triangle, the perpendiculars from the ends of the base to the opposite sides are equal.

88. If the bisector of the vertical angle of a triangle also bisects the base, the triangle is isosceles.

89. If the base AB of $\triangle ABC$ is produced to X, and if the bisectors of $\angle XBC$ and $\angle BAC$ meet at P, what fractional part is $\angle P$ of $\angle C$?

90. Given two parallels and a transversal, what angle do the bisectors of the interior angles on the same side of the transversal make with each other?

91. If one angle of an isosceles triangle is given, and it is known whether it is the vertical angle or not, then the other two angles are determined.

Proposition XXII.

96. Theorem. *The sum of the exterior angles of any polygon is a perigon.*

Given P and Q, two n-gons.

To prove that the sum of the exterior $\angle s = 360°$ in each n-gon.

Proof. 1. In P, each interior $\angle$ + its adjacent exterior $\angle = 180°$. § 14, def. st. $\angle$

2. $\therefore$ sum of int. and ext. $\angle s = n \cdot 180°$. Ax. 6

3. But sum of int. $\angle s = (n - 2) \cdot 180°$. Why?

4. $\therefore$ sum of ext. $\angle s = 2 \cdot 180° = 360°$. Ax. 3

The proof for Q is the same, if $\angle a$ is considered negative.

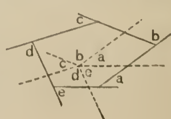
Exercises. 92. Each exterior angle of an equilateral triangle equals how many times each interior angle?

93. Each exterior angle of a regular heptagon equals what fractional part of each interior angle?

94. Each exterior angle of a regular n-gon equals what fractional part of each interior angle? See if the result found is true if $n = 3$, or 4.

95. Is it possible for the exterior angle of a regular polygon to be 70°? 72°? 75°? 120°?

96. Prove prop. XXII independently of prop. XXI by taking a point anywhere in the plane of the figure (inside or outside the polygon, or on the perimeter) and drawing parallels to the sides from that point, and showing that the sum of the exterior angles equals the perigon about that point.



97. Definitions. A quadrilateral whose opposite sides are parallel is called a **parallelogram.**

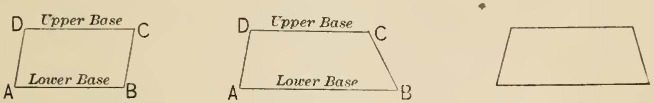
A quadrilateral that has one pair of opposite sides parallel is called a **trapezoid.**

Trapezium is a term often applied to a quadrilateral no two of whose sides are parallel.

By the definition of trapezoid here given it will be seen that the parallelogram may be considered a special form of the trapezoid.

The parallel sides of a trapezoid are called its *bases*, and are distinguished as *upper* and *lower.*

If the two opposite non-parallel sides of a trapezoid are equal, the trapezoid is said to be *isosceles.*



Parallelogram. Trapezoid. Isosceles trapezoid.

In the above figures, angles A, B, or B, C, or C, D, or D, A, are called *consecutive angles.* Angles A, C, or B, D, are called *opposite angles.*

Exercises. 97. If the student has proved ex. 96, let him prove prop. XXI from that.

98. Prove that the quadrilateral formed by the bisectors of the angles of any quadrilateral has its opposite angles supplemental.

99. Show that in ex. 98 the angles bisected may be either the four interior or the four exterior angles.

100. If from the ends of the base of an isosceles triangle perpendiculars are drawn to the opposite sides, a new isosceles triangle is formed, each of its base angles being half the vertical angle of the original triangle.

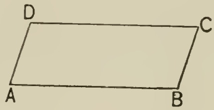
101. The hypotenuse is greater than either of the other sides of a right-angled triangle.

102. From the vertex of the right angle of a right-angled triangle, is it possible to draw, to the hypotenuse, a line longer than the hypotenuse? Proof.

103. A line from the vertex of an isosceles triangle to any point on the base produced is greater than either side. Is this also true for a scalene triangle?

PROPOSITION XXIII.

98. Theorem. *Any two consecutive angles of a parallelogram are supplemental, and any two opposite angles are equal.*



Given $\square ABCD$.

To prove that (1) $\angle A + \angle B = \text{st.} \angle$,

(2) $\angle A = \angle C$.

Proof. 1. $\angle A + \angle B = \text{st.} \angle$, which proves (1). Prop. XVII, cor. 2

2. $\angle B + \angle C = \text{st.} \angle$. Why?

3. $\therefore \angle A + \angle B = \angle B + \angle C$. Why?

4. $\therefore \angle A = \angle C$, which proves (2). Why?

COROLLARY. *If one angle of a parallelogram is a right angle, all of its angles are right angles.* (Why?)

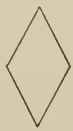
99. Definitions. If one angle of a parallelogram is a right angle, the parallelogram is called a **rectangle**.

By the corollary, all angles of a rectangle are right angles.

A parallelogram that has two adjacent sides equal is called a **rhombus**.

It is shown in prop. XXIV, cor. 1, that all of its sides are equal.

A rectangle that has two adjacent sides equal is called a **square**.



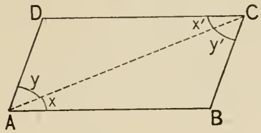
Rhombus.

Square.

It is shown in prop. XXIV, cor. 1, that all of its sides are equal. A square is thus seen to be a special form of a rhombus.

Proposition XXIV.

100. Theorem. *In any parallelogram, (1) either diagonal divides it into two congruent triangles, (2) the opposite sides are equal.*



Given $\square ABCD$.

To prove that (1) $\triangle ABC \cong \triangle CDA$,

(2) $AB = DC$.

Proof. 1. In the figure, $\angle x = \angle x'$, $\angle y = \angle y'$, and $AC \equiv AC$. Prop. (?)

2. $\therefore \triangle ABC \cong \triangle CDA$, which proves (1). Prop. II

3. $\therefore AB = DC$, which proves (2). § 57

Similarly for diagonal BD, and sides BC and AD.

Corollaries. 1. *If two adjacent sides of a parallelogram are equal, all of its sides are equal.*

For by step 3 the other sides are equal to these.

Hence, as stated in § 99, all of the sides of a rhombus are equal.

2. *The diagonals of a parallelogram bisect each other.*

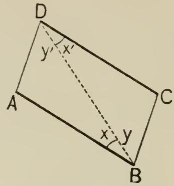
For if diagonal BD cuts AC at O, then, by prop. II, $\triangle ABO \cong \triangle CDO$, whence $AO = OC$, and $BO = OD$.

In the annexed figure, if a and a' are perpendicular to P and P', two parallels (prop. XVII, cor. 1), they are parallel (prop. XVI, cor. 3). Hence $a = a'$, by prop. XXIV. This fact is usually expressed by saying,

3. *Two parallel lines are everywhere equidistant from each other.*

Proposition XXV.

101. Theorem. *If a convex quadrilateral has two opposite sides equal and parallel, it is a parallelogram.*



Given a convex quadrilateral $ABCD$, with $AB = DC$, and $AB \parallel DC$.

To prove that $ABCD$ is a parallelogram.

Proof. 1. In the figure $\angle x = \angle x'$, Prop. (?)

$BD \equiv BD$, and $AB = DC$. Given

2. $\therefore \triangle ABD \cong \triangle CDB$, and $\angle y = \angle y'$. Prop. (?)

3. $\therefore BC \parallel AD$. Prop. XVI

4. $\therefore ABCD$ is a $\square$ by definition.

Exercises. 104. It is shown in Physics that if two forces are pulling from the point B, and the first force is represented (see fig. to prop. XXV) by BA, and the second by BC, the resultant (resulting force) will be represented by the diagonal BD. Show that, if the two forces do not pull in the same line, the resultant is always less than the sum of the two forces.

105. If two equal lines bisect each other at right angles, what figure is formed by joining the ends?

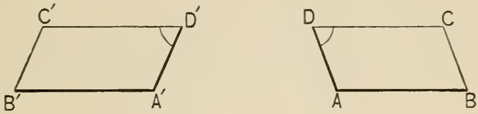
106. If the diagonals of a rectangle are perpendicular to each other, prove that the rectangle is a square.

107. On the diagonal BD of $\square$ $ABCD$, P and Q are so taken that $BP = QD$. Show that $APCQ$ is a parallelogram. Suppose P is on DB produced, and Q on BD produced.

108. Prove that the diagonals of a rectangle are equal. Prove that the diagonals of a rhombus are perpendicular to each other and bisect the angles of the rhombus.

PROPOSITION XXVI.

102. Theorem. *If two parallelograms have two adjacent sides and any angle of the one respectively equal to the corresponding parts of the other, they are congruent.*



Given ▱s $ABCD$, $A'B'C'D'$, in which $AB = A'B'$, $AD = A'D'$, and $\angle D = \angle D'$.

To prove that $\square ABCD \cong \square A'B'C'D'$.

Proof. 1. $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, for they are equal or supplemental to D or D'. Prop. XXIII

2. $CD = C'D'$, $BC = B'C'$, for they are equal to sides that are known to be equal. Prop. XXIV

3. Apply $\square ABCD$ to $\square A'B'C'D'$ so that AB coincides with its equal $A'B'$, A falling on A'. Then AD can be placed on $A'D'$ because $\angle A = \angle A'$. Then D will fall on D', because $AD = A'D'$. Similarly, C will fall on C', and CB on $C'B'$.

COROLLARIES. 1. *Two rectangles are congruent if two adjacent sides of the one are equal to any two adjacent sides of the other.* (Why?)

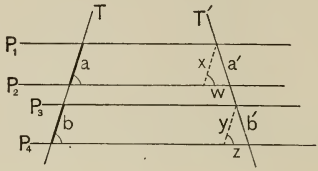
2. *Two squares are congruent if a side of the one equals a side of the other.* (Why?)

Exercises. 109. Is a parallelogram determined when any two sides and either diagonal are given? when two adjacent sides and either diagonal are given?

110. The angles at either base of an isosceles trapezoid are equal.

Proposition XXVII.

103. Theorem. *If there are two pairs of lines, all of which are parallel, and if the segments cut off by each pair on any transversal are equal, then the segments cut off on any other transversal are equal also.*



Given four parallels, of which P_1, P_2 cut off a segment a, and P_3, P_4 cut off an equal segment b, on a transversal T, and cut off segments a', b', respectively, on transversal T'.

To prove that $a' = b'$.

Proof. 1. Suppose x and $y \parallel T$ as in the figure.

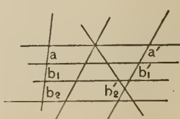
2. Then, in the figure, $\angle wx = \angle P_2T = \angle P_4T = \angle zy$. Prop. XVII, cor. 2

3. And $\angle a'w = \angle b'z$. Why?

4. And $x = a = b = y$. Prop. XXIV

5. $\therefore \triangle wa'x \cong \triangle zb'y$, and $a' = b'$. Prop. XIX, cor. 7

Corollaries. 1. *If a system of parallels cuts off equal segments on one transversal, it does on every transversal.*



For if $a = b_1$ or b_2, $a' = b_1'$ or b_2', respectively, and similarly for the other transversals.

2. *The line through the mid-point of one side of a triangle, parallel to another side, bisects the third side.*

Draw a third parallel through the vertex. Then cor. 1 proves it.

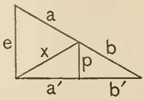
3. *The line joining the mid-points of two sides of a triangle is parallel to the third side.*

For if not, suppose through the mid-point of one of those sides a line is drawn parallel to the base; then this must bisect the other side, by cor. 2; $\therefore$ it must coincide with the line joining the mid-points, or else a side would be bisected at two different points. (This is the converse of cor. 2. Draw the figure.)

4. *The line joining the mid-points of two sides of a triangle equals half the third side.* (Prove it.)

Exercises. **111.** The line joining the mid-points of the non-parallel sides of a trapezoid is parallel to the bases.

112. In a right-angled triangle the mid-point of the hypotenuse is equidistant from the three vertices. (This exercise has been given before, and will be repeated, since it is important and admits of divers proofs. It is here easily proved by prop. XXVII, cor. 2; for if $a = b$, then $a' = b'$; but $p \parallel e$, $\therefore p \perp a'$, $\therefore x = b = a$.)



113. The lines joining the mid-points of the sides of a triangle divide it into four congruent triangles.

114. If one of the equal sides CB of an isosceles triangle ABC is produced through the base, and if a segment BD is laid off on the produced part, and an equal segment AE is laid off on the other equal side, then the line joining D and E is bisected by the base. (Consider the cases in which $BD < CB$, $BD = CB$, $BD > CB$.)

115. If the mid-points of the adjacent sides of any quadrilateral are joined, the figure thus formed is a parallelogram. (Consider this theorem for cases of concave, convex, and cross quadrilaterals, and for the special case of an interior angle of 180°.)

116. The lines joining the mid-points of the opposite sides of a quadrilateral bisect each other. Consider for the special cases mentioned in ex. 115.

117. The line joining the mid-points of the diagonals of a quadrilateral, and the lines joining the mid-points of its opposite sides, pass through the same point.

118. P and Q are the mid-points of the sides AB and CD of the parallelogram $ABCD$. Prove that PD and BQ trisect (divide into three equal segments) the diagonal AC.

EXERCISES.

119. What is the sum of the interior angles of a polygon of 20 sides? of 30 sides?

120. How many degrees in each angle of a regular polygon of 12 sides? of 20 sides?

121. How many sides has a polygon the sum of whose interior angles is 48 right angles?

122. The vertical angle of a certain isosceles triangle is $11° 15' 20''$; how large are the base angles?

123. The exterior angle of a certain triangle is 140°, and one of the interior non-adjacent angles is a right angle; how many degrees in each of the other two interior angles?

124. Each exterior angle of a certain regular polygon is 10°; how many sides has the polygon?

125. If P is any point on the side BC of $\triangle ABC$, then the greater of the sides AB, AC, is greater than AP.

126. If the diagonals of a quadrilateral bisect each other, prove that the quadrilateral is a parallelogram. Of what corollary is this the converse? Prove that the diagonals of an isosceles trapezoid are equal.

127. Conversely, prove that if the diagonals of a trapezoid are equal, the trapezoid is isosceles.

128. Is a parallelogram determined when its two diagonals are given? when its two diagonals and their angle are given?

129. ABC is a triangle; AC is bisected at M; BM is bisected at N; AN meets BC at P; MQ is drawn parallel to AP to meet BC at Q. Prove that BC is trisected (see ex. 118) by P and Q.

130. A, C are points on the same side of XX'; B is the mid-point of AC; through A, B, C parallels are drawn cutting XX' in A', B', C'. Prove that $AA' + CC' = 2\,BB'$.

131. A straight line drawn perpendicular to the base AB of an isosceles triangle ABC cuts the side CA at D and BC produced at E; prove that CED is an isosceles triangle.

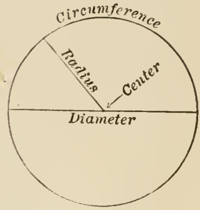
132. ABC is a triangle, and the exterior angles at B and C are bisected by the straight lines BD, CD respectively, meeting at D; prove that $\angle CDB + \frac{1}{2}\angle A =$ a right angle.

133. In the triangle ABC the side BC is bisected at E, and AB at G; AE is produced to F so that $EF = AE$, and CG is produced to H so that $GH = CG$. Prove that F, B, H are in one straight line.

3. PROBLEMS.

104. Definitions. A **curve** is a line no part of which is straight.

105. A **circle** is the finite portion of a plane bounded by a curve, which is called the **circumference**, and is such that all points on that line are equidistant from a point within the figure called the **center** of the circle.



A circle is evidently described by a line-segment making a complete rotation in a plane, about a fixed point (the center).

106. A straight line terminated by the center and the circumference is called a **radius**, and a straight line through the center terminated both ways by the circumference is called a **diameter** of the circle.

107. A part of a circumference is called an **arc**.

NOTE. The above definitions are substantially those usually met in elementary geometries. The student will find, after leaving this subject, that the word *circle* is often used for *circumference*. Indeed, there is good authority for so using the word even in elementary geometry.

108. From the above definitions the following *corollaries* may be accepted without further proof:

1. *A diameter of a circle is equal to the sum of two radii of that circle.*

2. *Circles having the same radii are congruent.*

3. *A point is within a circle, on its circumference, or outside the circle, according as the distance from that point to the center is less than, equal to, or greater than, the radius.*

109. It now becomes necessary to assume certain postulates relating to the circle.

Postulates of the Circle.

1. *All radii of the same circle are equal, and hence all diameters of the same circle are equal.*

2. *If an unlimited straight line passes through a point within a circle, it must cut the circumference at least twice, and so for any closed figure.*

That it cannot cut the circumference more than twice is proved in III, prop. VI, cor.

3. *If one circumference intersects another once, it intersects it again.*

4. *A circle has but one center.*

5. *A circle may be constructed with any center, and with a radius equal to any given line-segment.*

This postulate requires the use of the compasses. As has been stated, *the only instruments allowed in elementary geometry are the compasses and the straight-edge*, a limitation due to Plato. In the more advanced geometry, where other curves than the circle are studied, other instruments are permitted.

110. Order to be observed in the solution of problems:

Given. For example, the angle A.

Required. For example, to bisect that angle.

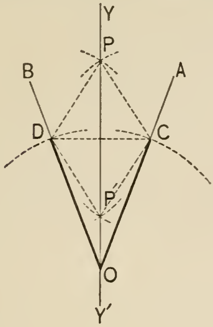
Construction. A statement of the process of solving, using only the straight-edge and compasses in drawing the figure described.

Proof. A proof that the construction has fulfilled the requirements.

Discussion. Any consideration of special cases, of the limitations of the problem, etc. If a problem has but a single solution, as that an angle may be bisected but once, the solution is said to be *unique*.

PROPOSITION XXVIII.

111. Problem. *To bisect a given angle.*



Given the $\angle AOB$.

Required to bisect it.

Construction. 1. With center O describe an arc cutting AO at C, and OB at D. § 109, post. of ⊙

2. Draw DC. § 28, post. st. line

3. Describe arcs with centers D, C, and radius DC. Post. (?)

4. Join their intersection P, with O. Post. (?)

Then $\angle AOB$ is bisected, YY' being an axis of symmetry (§ 68).

Proof. 1. Draw DP, CP;

then $OD = OC$, $DP = DC$, $DC = CP$. § 109, 1

2. $\therefore DP = CP$. Ax. (?)

3. But $OP \equiv OP$.

4. $\therefore \triangle OCP \cong \triangle ODP$,

and $\angle COP = \angle POD$. Prop. XII

COROLLARY. *An angle may be divided into 2, 4, 8, 16, ... 2^n, equal angles.* (How ?)

112. Note on Assumed Constructions. It has been assumed, up to prop. XXVIII, that all constructions were made as required for the theorems. Thus an equilateral triangle has been frequently mentioned, although the method of constructing one has not yet been indicated; a regular heptagon has been mentioned in ex. 93, and reference might be made to certain results following from the trisection of an angle, although the solutions of the problems, to construct a regular heptagon, and to trisect any angle, are impossible by elementary geometry. But the possibility of solving such problems has nothing to do with the logical sequence of the theorems; one may know that each angle of a regular heptagon is $\frac{5}{7} \cdot 180°$, whether the regular heptagon admits of construction or not. Nevertheless, an important part of geometry concerns itself with the construction of certain figures — a part of utmost practical value and of much interest to the student of mathematics.

113. Suggestions on the Solution of Problems. The methods of logically undertaking the solution of problems will be discussed at the close of Book III. But at present one method, already suggested on p. 35, should be repeated: *In attempting the solution of a problem, assume that the solution has been accomplished; then analyze the figure and see what results follow; then reverse the process, making these results precede the solution.*

For example, in prop. XXVIII, *assume that $\angle AOB$ has been bisected by YY'*; *if that were done*, and if any point, P, on YY' were joined to points equidistant from O, on the arms, say C and D, then $\triangle OCP$ would be congruent to $\triangle ODP$; *now reverse the process* and attempt to make $\triangle OCP$ congruent to $\triangle ODP$; this can be done if OD can be made equal to OC, and PD to PC, because $OP \equiv OP$; but this can be done by § 109, 5.

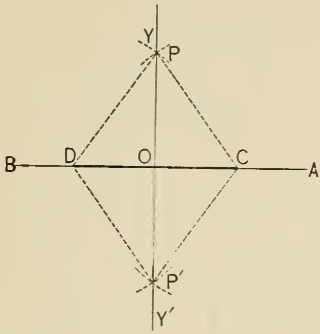
This method of attacking a problem, without which the student will grope in the dark, is called **Geometrical Analysis.**

Exercises. 134. Give the solution of prop. XXVIII, using P' instead of P. Why is P better than P' for practical purposes? In what case would the construction fail for the point P'? In that case how many degrees in $\angle AOB$?

135. In prop. XXVIII, in what case would P' fall below O? Give the solution in that case, after connecting P' and O and producing $P'O$.

PROPOSITION XXIX.

114. Problem. *To draw a perpendicular to a given line from a given internal point.*



Solution. This is merely a special case of prop. XXVIII, the case in which $\angle AOB$ is a straight angle. (Why?) The construction and proof are identical with those of prop. XXVIII, and the student should give them to satisfy himself of this fact.

Exercises. 136. What kind of a quadrilateral is $CPDP'$? Prove it.

137. Prove that any point on BA is equidistant from P and P'. Also that any point on YY' is equidistant from D and C.

138. In step 3 of the construction of prop. XXVIII might the radius equal two times DC? If so, complete the solution. Is there any limit to the length of the radius in that step?

139. In the figure of prop. XXVIII, suppose $\angle PCO = 130°$. Find the number of degrees in the various other angles, not reflex, of the figure.

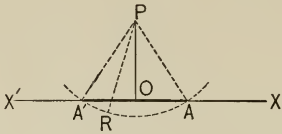
140. In the figure of prop. XXVIII, prove that the reflex angle BOA is bisected by YY', that is, by PO produced.

141. Also prove that YY' is the perpendicular bisector of DC.

142. Also prove that if O is connected with P and with P', OP' will fall on OP. (Prel. prop. VIII.)

Proposition XXX.

115. Problem. *To draw a perpendicular to a given line from a given external point.*



Given the line XX' and the external pt. P.

Required to draw a perpendicular from P to XX'.

Construction. 1. Draw PR cutting XX'. § 28

2. With center P and radius PR const. a ⊙. § 109, post. of ⊙

3. Join A and A', where the circumference cuts XX', with P. § 28, post. of st. line

4. Bisect $\angle A'PA$. Prop. XXVIII

The bisector, PO, is the required perpendicular.

Proof. 1. $PA = PA'$, § 109, 1

$\angle OPA = \angle A'PO$, Const., 4

and $PO \equiv PO$.

2. $\therefore \triangle APO \cong \triangle A'PO$, Prop. I

and $\angle AOP = \angle POA'$.

3. $\therefore \angle AOP$ is a rt. $\angle$, and $PO \perp XX'$. §§ 19, 20

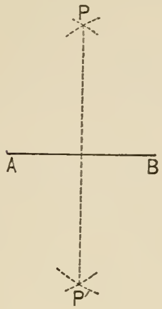
Note. The solution of this problem is attributed to Œnopides.

Exercises. **143.** Find in a given line a point equidistant from two given points A and B, the mid-point of AB being also given.

144. Find a point equidistant from three given points A, B, C, the mid-points of AB and BC being also given.

PROPOSITION XXXI.

116. Problem. *To bisect a given line.*



Given the line AB.

Required to bisect it.

Construction. 1. With centers A, B, and equal radii describe arcs intersecting at P and P'. Post. (?)

2. Draw PP'. Post. (?)

3. Then PP' bisects AB.

Proof.

(Let the student give it. Draw AP', $P'B$, BP, PA.)

Exercises. **145.** Through a given point to draw a line making equal angles with the arms of a given angle. Discuss for various relative positions of the point.

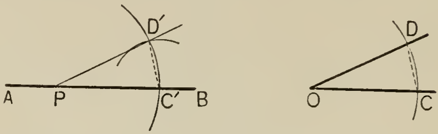
146. To draw a perpendicular to a line from one of its extremities, when the line cannot be produced. (Ex. 112 suggests a plan.)

147. Through two given points on opposite sides of a given line draw two lines which shall meet in the given line and include an angle which is bisected by that line.

148. If two isosceles triangles have a common base, the straight line through their vertices is a perpendicular bisector of the base.

Proposition XXXII.

117. Problem. *From a given point in a given line to draw a line making with the given line a given angle.*



Given the line AB, the point P in it, and the angle O.

Required from P to draw a line making with AB an angle equal to $\angle O$.

Construction. 1. On the arms of $\angle O$ lay off $OC = OD$ by describing an arc with center O and any radius OC. § 109, post. of ⊙

2. Draw CD. § 28, post. of st. line

3. With center P and radius OC, describe a circumference cutting PB in C'. Post. (?)

4. With center C' and radius CD, describe an arc cutting the circumference in D'. Post. (?)
Draw PD', and this is the required line.

Proof. Draw $C'D'$; then,
$\triangle PC'D'$ and $\triangle OCD$ being mutually equilateral, Why?

$$\triangle PC'D' \cong \triangle OCD,$$

and $\angle C'PD' = \angle COD$. Prop. XII

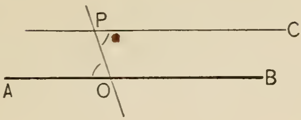
Exercises. 149. Prove that the circumferences must cut at D' as stated in step 4.

150. See if the solution of prop. XXXII is general enough to cover the cases where the $\angle O$ is straight, reflex, a perigon.

151. From a given point in a given line to draw a line making an angle supplemental to a given angle.

PROPOSITION XXXIII.

118. Problem. *Through a given point to draw a line parallel to a given line.*



Given the line AB and the point P.

Required through P to draw a line parallel to AB.

Construction. 1. Join P with any point, O, on AB. § 28, post. of st. line

2. From P draw PC making $\angle OPC = \angle POA$. (?)
Then PC is the required line.

Proof. $PC \parallel AB$. Why ?

Discussion. The solution fails if P is on the unlimited line AB.

Exercises. 152. Through a given point to draw a line making a given angle with a given line. Notice that the solution is not unique.

153. Through a given point to draw a transversal of two parallels, from which the parallels shall cut off a given segment. Discussion should show when there are two solutions, when only one, when none.

154. To construct a polygon (say a hexagon) congruent to a given polygon.

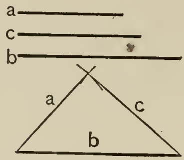
155. Through two given points to draw two lines forming with a given unlimited line an equilateral triangle.

156. Three given lines meet in a point; draw a transversal such that the two segments of it, intercepted between the given lines, may be equal. Is the solution unique ?

157. From P, the intersection of the bisectors of two angles of an equilateral triangle, draw parallels to two sides of the triangle, and show that these parallels trisect (see ex. 118) the third side.

Proposition XXXIV.

119. Problem. *To construct a triangle, given the three sides.*



Given a, b, c, three sides of a triangle.

Required to construct the triangle.

Construction. 1. With the ends of b as centers, and with radii a, c, describe circumferences. § 109

2. Connect either point of intersection of these circumferences with the ends of b. § 28

Then is the required △ constructed.

Proof. It was constructed on b, and the other sides equal a, c. § 109, 1

Discussion. If the two circumferences do not intersect, a solution is impossible, for then either $a > b + c$, $a = b + c$, $a = c - b$, or $a < c - b$, and in none of these cases is a triangle possible.

Prop. VIII and cor.

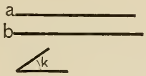
Corollary. *To construct an equilateral triangle on a given line-segment.*

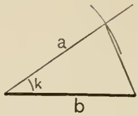
The first proposition of Euclid's "Elements of Geometry." Euclid proceeded upon the principle of logical sequence of propositions, with no attempt at grouping the theorems and the problems separately. He found this corollary (a problem) the best proposition with which to begin his system.

Exercise. 158. In a given triangle inscribe a rhombus, having one of its angles coincident with a given angle of the triangle, and the other three vertices on the three sides of the triangle.

PROPOSITION XXXV.

120. Problem. *To construct a triangle, given two sides and the included angle.*





Given the sides a, b, and the included angle k.

Required to construct the triangle.

Construction. 1. From either end of b draw a line making with b the angle k. Prop. XXXII

2. On that line mark off a by describing an arc of radius a. § 109

3. Join the point thus determined with the other end of b. § 28

Then the triangle is constructed.

Proof. By step 2 the line marked off equals a, and by step 1 $\angle b = \angle k$, and it is constructed on b.

Exercises. 159. To trisect a right angle. (Construct an equilateral triangle on one arm.)

160. On the side AC of $\triangle ABC$ to find the point P such that the parallel to AB, from P, meeting BC at D, shall have $PD = AP$.

161. To construct a triangle, having given two angles and the perpendicular from the vertex of the third angle to the opposite side.

162. Draw a line parallel to a given line, so that the segment intercepted between two other given lines may equal a given segment. 4/11

163. Given the three mid-points of the sides of a triangle, to construct the triangle.

164. Through a given point P in an angle AOB to draw a line, terminated by OA and OB, and bisected at P. (Through P draw a $\parallel$ to BO cutting OA in X; on XA lay off $XY = OX$; draw YP.)

10/11-05

Proposition XXXVI.

121. Problem. *To construct a triangle, given two sides and the angle opposite one of them.*

Given two sides of a triangle, a, b, and $\angle k$ opposite a.

Required to construct the triangle.

Construction. 1. At either end of b draw a line making with b an angle equal to $\angle k$. Prop. (?)

2. With the other end of b as a center, and a radius a, describe a circumference. Post. (?)

3. Join the points where the circumference cuts the line of step 1, with the center. Post. (?)

Then the triangle is constructed.

Proof. For it has the given side b, and the given $\angle k$, and the lines of step 3 equal a. § 109, 1

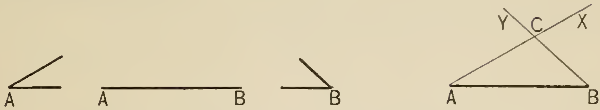
Discussion. If the circumference cuts the line twice, two solutions are possible, and the triangle is ambiguous (see prop. XIV). If it touches the line without cutting it, what about the solution? If it does not meet the line, no solution is possible. If $\angle k$ is right or obtuse, or if $a \nless b$, only one solution is possible (prop. XIX, cor. 5).

Draw a figure for each of these cases, and show from the drawings that the statements made in the discussion are true.

Exercise. 165. XX', YY', are two given lines through O, and P is a given point; through P to draw a line to XX', which shall be bisected by YY'. Investigate for various positions of P, as where P is within the $\angle XOY$, the $\angle YOX'$, on OY, or on OX.

PROPOSITION XXXVII.

122. Problem. *To construct a triangle, given two angles and the included side.*



Given two angles, A, B, and the included side AB.

Required to construct the triangle.

Construction. 1. From A draw AX making with AB an angle equal to $\angle A$. Prop. XXXII

2. Similarly, from B draw BY, making an angle equal to $\angle B$. Prop. XXXII

C being the intersection of AX, BY, then ABC is the required $\triangle$.

Proof. (Let the student give it.)

Discussion. If AX, BY, do not intersect, what follows?

PROPOSITION XXXVIII.

123. Problem. *To construct a triangle, given two angles and a side opposite one of them.*

Solution. Subtract the sum of the angles (found by prop. XXXII) from 180° and thus find the third angle (prop. XIX). The problem then reduces to prop. XXXVII.

PROPOSITION XXXIX.

124. Problem. *To construct a square on a given line as a side.*

4. LOCI OF POINTS.

125. The place of all points satisfying a given condition is called the **locus of points** satisfying that condition.

Indeed, the word *locus* (Latin) means simply *place* (English, *locality*, *locate*, etc.) ; the plural is *loci.*

For example, if points are on this page and are one inch from the left edge, their locus is evidently a straight line parallel to the edge.

Furthermore, the locus of points at a given distance r from a fixed point O is the circumference described about O with a radius r. This statement, although very evident, is made a theorem (prop. XL) because of the frequent reference to it.

Of course in this discussion, as elsewhere in Books I–V, the points are all supposed to be confined to one plane.

In Plane Geometry the loci considered will be found to consist of one or more straight or curved lines.

It is a common mistake to assume that a locus, which one is trying to discover, consists of a single line. It may consist of two lines, as in prop. XLII.

126. In proving a theorem concerning the locus of points it is necessary and sufficient to prove two things:

1. *That any point on the supposed locus satisfies the condition;*

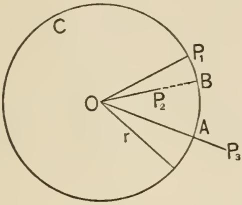
2. *That any point not on the supposed locus does not satisfy the condition.*

For if only the first were proved, there might be some other line in the locus; and if only the second were proved, the supposed locus might not be the correct one.

Exercise. 166. State, without proof, what is (1) the locus of points $\frac{1}{2}$ in. from a given straight line; (2) the locus of points equidistant from two parallel lines.

PROPOSITION XL.

127. Theorem. *The locus of points at a given distance from a given point is the circumference described about that point as a center, with a radius equal to the given distance.*



Given the point O, the line r, and the circumference C described about O with radius r.

To prove that C is the locus of points r distant from O.

Proof. 1. Let P_1, P_2, P_3 be points on C, within the circle, and without the circle, respectively.
Let OP_2 produced meet C in B, and OP_3 meet C in A.

2. Then $OP_1 = OB = OA = r$, § 109, 1
and $OP_2 < OB$, and $OP_3 > OA$. Ax. 8

3. $\therefore$ any point on C is r distant from O, and any point not on C is not r distant from O.

Exercises. 167. Has it been proved in prop. XL that the required locus may not be merely the arc cut off by r and OP_1? If so, where?

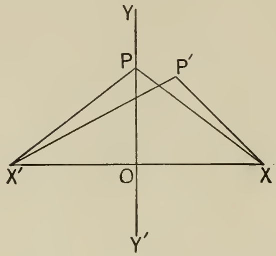
168. What is the locus of points at a distance of $\frac{1}{4}$ in. from the above circumference, the distance being measured on a line through O?

169. Lighthouses on two islands are 10 miles apart; show that there are two points at sea which are exactly 12 miles from each.

170. How would you find, by the intersection of two loci, a point on this page 1 in. from O in the above figure, and 3 in. from the right edge of the paper?

Proposition XLI.

128. Theorem. *The locus of points equidistant from two given points is the perpendicular bisector of the line joining them.*



Given two points X and X', and $YY' \perp XX'$ at the mid-point O.

To prove that YY' is the locus of points equidistant from X and X'.

Proof. 1. Let P be any point on YY', and P' be any point not on YY'.

Draw PX, PX', $P'X$, $P'X'$.

2. Then $PX = PX'$, Prop. XX, cor. 5

and $P'X' > P'X$. Prop. XX, cor. 2

3. Hence any point on YY' is equidistant from X and X', but any point not on YY' is unequally distant from X and X'.

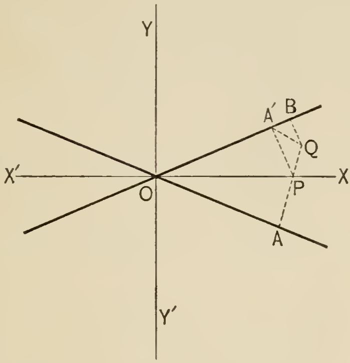
$\therefore$ YY' is the locus. § 125, def.

Exercises. 171. Required to find a point which is 1 in. from X and $\frac{3}{4}$ in. from X' in the above figure. Is there more than one such point?

172. Required to find a point which is equidistant from X and X' in the above figure, and 1 in. from O. Is there more than one such point?

PROPOSITION XLII.

129. Theorem. *The locus of points equidistant from two given lines consists of the bisectors of their included angles.*



Given OA and OB, two lines intersecting at O, and XX' and YY' the bisectors of the angles at O.

To prove that XX' and YY' form the locus of points equidistant from OA and OB.

Proof. 1. Let Q be any point on neither XX' nor YY'; let $QB \perp OB$, $QA \perp OA$, QA cut OX in P, $PA' \perp OB$. Draw QA'.
Since Q may be moved, P may be considered as any point on OX.

2. Then $\triangle OAP \cong \triangle OA'P$,
and $AP = A'P$. Prop. XIX, cor. 7

3. Also, $A'P + PQ > A'Q > BQ$. Why?

4. $\therefore AQ$, or $AP + PQ > BQ$. Why?

5. $\therefore$ any point P on XX' (or on YY') is equidistant from OA and OB, but any point Q on neither XX' nor YY' is unequally distant from OA and OB.

Corollaries. 1. *If the given lines are parallel, the locus is a parallel midway between them.* (Prove it.)

The student should imagine the effect of keeping points A, A' fixed, and moving O farther to the left. YY' moves with O, but XX' keeps its position as the lines approach the condition of being parallel.

2. *The locus of points at a given distance from a given line consists of a pair of parallels at that distance, one on each side of the fixed line.* (Prove it.)

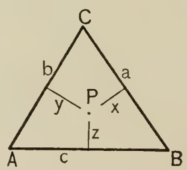
130. Definitions.

Three or more *lines* which meet in a *point* are said to be **concurrent**.

Three or more *points* which lie in a *line* are said to be **collinear**.

Proposition XLIII.

131. Theorem. *The perpendicular bisectors of the three sides of a triangle are concurrent.*



Given a triangle of sides a, b, c, and x, y, z their respective perpendicular bisectors.

To prove that x, y, z are concurrent.

Proof. 1. x and y must meet as at P. Prop. XVII, cor. 4

2. Then P is equidistant from B and C, and C and A. Prop. XLI

3. $\therefore$ P is on the perpendicular bisector of c; Why?

i.e. z passes through P.

COROLLARIES. 1. *The point equidistant from three non-collinear points is the intersection of the perpendicular bisectors of any two of the lines joining them.*

Step 2.

2. *There is one circle, and only one, whose circumference passes through three non-collinear points.*

Let A, B, C be the three points. Then by step 2 they are equidistant from P, the intersection of x and y.

And $\because$ x and y contain all points equidistant from A, B, and C, and can intersect but once, there is only one point P.

And $\because$ there is only one center and one radius, there is one and only one circle.

3. *Circumferences having three points in common are identical.*

Otherwise cor. 2 would be violated.

4. *If from a point more than two lines to a circumference are equal, that point is the center of the circle.*

For suppose a circumference through A, B, C, and suppose $PA = PB = PC$.

Now with center P and radius PA a circumference can be described through A, B, C, because it is given that

$$PA = PB = PC. \qquad \text{§ 108, cor. 3}$$

And this is identical with the given circumference.
Prop. XLIII, cor. 3

$\therefore$ its center must be identical with the given center, since a $\odot$ cannot have two centers. § 109, 4

Exercises. 173. The proof of prop. XLIII is, of course, the same if the triangle is right-angled or obtuse-angled. The figures, however, show interesting positions for P; consider them.

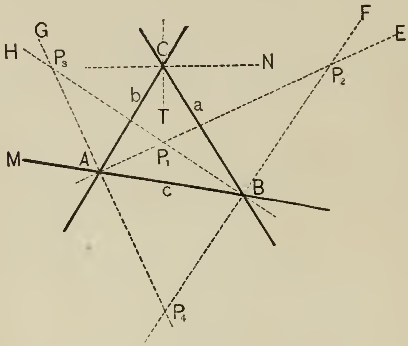
174. Required to find a point at a given distance d from a fixed point O, and equidistant from two given intersecting lines. How many such points can be found in general?

175. Required to find a point equidistant from two given intersecting lines, and equidistant from two given points. How many such points can be found in general?

14/11

Proposition XLIV.

132. Theorem. *The bisectors of the interior and exterior angles of a triangle are concurrent four times by threes.*



Given the $\triangle ABC$, and the bisectors of the interior and exterior angles, lettered as in the figure.

To prove that these bisectors are concurrent four times by threes; that is, 3 meet at P_1, 3 at P_2, etc.

Proof. 1. $\because \angle CAM > \angle CBM, \therefore \angle GAM > \angle HBM$. Prop. V

2. $\therefore AG$ and BH meet as at P_3. Prop. XVII, cor. 3

3. $\angle HBM + \angle BAE < \angle B + \angle A < 180°$. Prop. XIX

$\therefore BH$ and AE meet as at P_1. Prop. XVII, cor. 3

4. $BF \perp BH$, and $AG \perp AE$, Prel. prop. IX

$\therefore BF$ and AG meet as at P_4. Prop. XVII, cor. 4

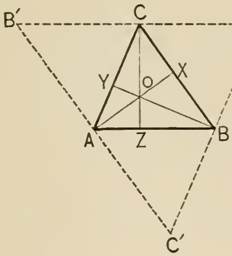
5. Also, P_1 is equidistant from a and c, from c and b, and $\therefore$ from a and b, Prop. XLII

$\therefore P_1$ lies on CT. Similarly for P_4. Prop. XLII

6. Similarly, P_2 and P_3 lie on CN. $\therefore$ the four points P_1, P_2, P_3, P_4, are points of concurrence of the bisectors.

PROPOSITION XLV.

133. Theorem. *The perpendiculars from the vertices of a triangle to the opposite sides are concurrent.*



Given the $\triangle ABC$.

To prove that the perpendiculars from A, B, C, to a, b, c, respectively, are concurrent.

Proof. 1. Through A, B, C, respectively, suppose $B'C' \parallel CB$, $A'C' \parallel CA$, $A'B' \parallel BA$.

2. Then $ABCB'$ and $ABA'C$ are $\boxed{s}$. Def. $\square$

3. $\therefore B'C = AB = CA'$, and C is the mid-point of $B'A'$. Prop. XXIV; ax. 1

4. Similarly, A and B are mid-points of $B'C'$, $C'A'$.

5. If $AX, BY, CZ \perp B'C', C'A', A'B'$, respectively, they are concurrent, as at O. Prop. XLIII

6. And they are also the perpendiculars from A, B, C to a, b, c. Prop. XVII, cor. 1

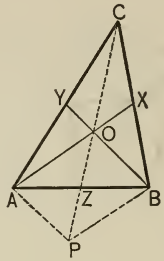
NOTE. The theorem is due to Archimedes.

134. Definition. To **trisect** a magnitude is to cut it into three equal parts.

Exercise. 176. In prop. XLIV suppose C moves down to the side c. What becomes of P_1, P_2, P_3, P_4?

Proposition XLVI.

135. Theorem. *The medians of a triangle are concurrent in a trisection point of each.*



Given the $\triangle ABC$ and the medians BY, AX, intersecting at O.

To prove that (1) the median from C must pass through O,
(2) $OX = \frac{1}{3} AX$, $OY = \frac{1}{3} BY$, etc.

Proof. 1. Suppose CO drawn, and produced indefinitely, cutting AB at Z.

2. Suppose $AP \parallel OB$; CO must cut AP, as at P. § 85

3. Draw PB. Then $\because CY = YA$, $\therefore CO = OP$.
Prop. XXVII, cor. 2

4. And $\because CO = OP$, and $CX = XB$, $\therefore OX \parallel PB$.
Prop. XXVII, cor. 3

5. $\therefore APBO$ is a $\square$, $AZ = ZB$, and $OZ = ZP$.
Prop. XXIV, cor. 2

6. $\therefore CZ$ is a median, and it passes through O.

7. And $\because OZ = \frac{1}{2} OP$, $\therefore OZ = \frac{1}{2} CO$, or $\frac{1}{3} CZ$. Similarly for OY and OX.

Exercise. 177. The sum of the three medians of a triangle is greater than three-fourths of its perimeter.

136. Definitions. The point of concurrence of the perpendicular bisectors of the sides of a triangle is called the **circumcenter** of the triangle. (Prop. XLIII.)

The reason will appear later when it is shown that this point is the *center* of the *circum*-scribed circle. (See Table of Etymologies.)

137. The point of concurrence of the bisectors of the *interior* angles of a triangle is called the **in-center** of the triangle; the points of concurrence of the bisectors of two *exterior* angles and one interior are called the **ex-centers** of the triangle. (Prop. XLIV.)

It will presently be proved that the *in*-center is the center of a circle, *in*-side the triangle, just touching the sides; and that the *ex*-centers are centers of circles, *out*-side the triangle, just touching the three lines of which the sides of the triangle are segments. Hence the names *in-center* and *ex-center*.

138. The point of concurrence of the three perpendiculars from the vertices to the opposite sides is called the **orthocenter** of the triangle. (Prop. XLV.)

139. The point of concurrence of the three medians of a triangle is called the **centroid** of that triangle. (Prop. XLVI.)

It is shown in Physics that this point is also the center of mass, or center of gravity of the plane surface of the triangle. It is, therefore, sometimes called by those names.

Exercises. 178. If a triangle is acute-angled, prove that both the circumcenter and the orthocenter lie within the triangle.

179. In prop. XLVI, if X, Y, Z be joined, prove that the $\triangle XYZ$ will be equiangular with the $\triangle ABC$.

180. Is there any kind of a triangle in which the in-center, circumcenter, orthocenter, and centroid coincide? If so, what is it? Prove it.

181. In the figure of prop. XLVI, connect X, Y, Z, and prove that O is also the centroid of $\triangle XYZ$.

182. In ex. 179, prove that if the mid-points of the sides of $\triangle XYZ$ are joined, O is also the centroid of that triangle; and so on.

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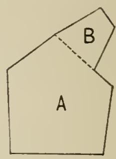
BOOK II. — EQUALITY OF POLYGONS.

1. THEOREMS.

140. Definitions. Two polygons are said to be **adjacent** if they have a segment of their perimeters in common.

141. Suppressing the common segment of the perimeters of two adjacent polygons, a polygon results which is called the **sum** of the two polygons. Similarly for the *sum* of several polygons, and for the **difference** of two overlapping polygons.

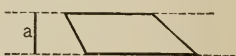


142. Surfaces which may be divided into the same number of parts respectively congruent, or which are the differences between congruent surfaces, are said to be **equal**.

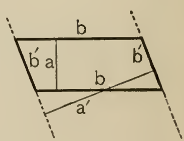
This property is often designated by the expressions *equivalent*, *equal in area*, *of equal content*, etc.; but the use of the word *congruent*, for *identically equal*, renders the word *equal* sufficient.

The definition is more broadly treated in Book V.

143. The **altitude of a trapezoid** is the perpendicular distance between the base lines.

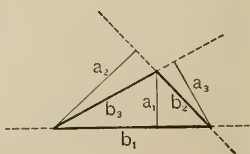


Hence a trapezoid can have but one altitude, a, unless it becomes a parallelogram.



144. The **altitude of a triangle** with reference to a given side as the base, is the distance from the opposite vertex to the base line.

Hence a triangle can have three distinct altitudes, viz. a_1, a_2, a_3, in the figure.



PROPOSITION I.

145. Theorem. *Parallelograms on the same base or on equal bases and between the same parallels are equal.*

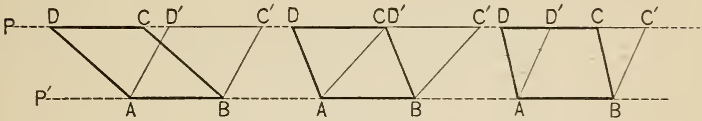


FIG. 1. FIG. 2. FIG. 3.

Given ▱s $ABCD$, $ABC'D'$, on the same base AB, and between the same parallels P, P'.

To prove that ▱ $ABCD$ = ▱ $ABC'D'$.

Proof. 1. $AD = BC$, $AD' = BC'$, $DC = AB = D'C'$. Why?

2. In Fig. 1, adding CD', $DD' = CC'$. Ax. 2

3. $\therefore \triangle BC'C \cong \triangle AD'D$. Why?

4. But $ABC'D \equiv ABC'D$.

$\therefore$ ▱ $ABCD$ = ▱ $ABC'D'$. Ax. 3

Similarly for Figs. 2 and 3. In Fig. 2, CD' has become zero; in Fig. 3, it has become negative.

The meaning of "between the same parallels" is apparent.

COROLLARIES. 1. *A parallelogram equals a rectangle of the same base and the same altitude.* (Why?)

2. *Parallelograms having equal bases and equal altitudes are equal.* (Why?)

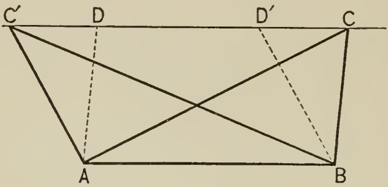
3. *Of two parallelograms having equal altitudes, that is the greater which has the greater base; and of two having equal bases, that is the greater which has the greater altitude.* (Why?)

4. *Equal parallelograms on the same base or on equal bases have equal altitudes.*

Law of Converse, § 73, after cors. 2 and 3. Give it in full.

Proposition II.

146. Theorem. *Triangles on the same base or on equal bases and between the same parallels are equal.*



Given $\triangle\!\!\triangle$ ABC, ABC' on the base AB, and between the same parallels AB, $C'C$.

To prove that $\triangle ABC = \triangle ABC'$.

Proof. 1. In the figure, suppose $AD \parallel BC$, $BD' \parallel AC'$.

Then $ABCD$, $ABD'C'$ are equal ▱s. Why?

2. And, since $\triangle\!\!\triangle$ ABC, ABC' are their halves, I, prop. XXIV

$\therefore \triangle ABC = \triangle ABC'$. Ax. 7

Corollaries. 1. *A triangle equals half of a parallelogram, or half of a rectangle, of the same base and the same altitude as the triangle.*

By step 2, and prop. I, cor. 1.

2. *Triangles having equal bases and equal altitudes are equal.*

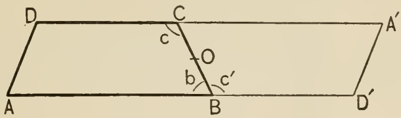
3. *Of two triangles having equal altitudes, that is the greater which has the greater base; and of two having equal bases, that is the greater which has the greater altitude.* (Why?)

4. *Equal triangles on the same base or on equal bases have equal altitudes.* (Why?)

Note. In props. I and II if the figures are on *equal* bases they can evidently be placed on the same base. Hence the proofs given are sufficient.

PROPOSITION III.

147. Theorem. *A trapezoid is equal to half of the rectangle whose base is the sum of the two parallel sides, and whose altitude is the altitude of the trapezoid.*



Given the trapezoid $ABCD$.

To prove that $ABCD$ equals half of a rectangle with the same altitude, and with base equal to $AB + DC$.

Proof. 1. About O, the mid-point of BC, revolve $ABCD$ through $180°$ to the position $A'CBD'$, leaving its original trace.

2. Then, $\because \angle c' \equiv \angle c$, and $\angle b + \angle c =$ st. $\angle$,

$\therefore \angle b + \angle c' = \angle b + \angle c =$ st. $\angle$,

and $\therefore ABD'$ is a st. line. § 14, def. st. $\angle$

Similarly, DCA' is a st. line.

3. Also, $\because \angle D' \equiv \angle D$, and $\angle A + \angle D =$ st. $\angle$,

$\therefore \angle A + \angle D' = \angle A + \angle D =$ st. $\angle$,

$\therefore D'A' \parallel AD$, I, prop. XVI, cor. 2

and $\therefore AD'A'D$ is a $\square$. § 97, def. $\square$

4. The base of the $\square = AB + DC$,

and the $\square = 2 \cdot ABCD$. Why?

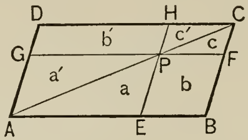
5. $\therefore ABCD = \frac{1}{2} \square = \frac{1}{2}$ required $\square$. Prop. I, cor. 1

Exercises. 183. P is any point within $\square$ $ABCD$. Prove that $\triangle PAB + \triangle PCD = \frac{1}{2} \square ABCD$. Suppose P is outside of $\square$ $ABCD$.

184. A quadrilateral equals a triangle of which two sides equal the diagonals of the quadrilateral, and the included angle of those sides equals the included angle of the diagonals.

Proposition IV.

148. Theorem. *If through a point on a diagonal of a parallelogram parallels to the sides are drawn, the parallelograms on opposite sides of that diagonal are equal.*



Given $\square ABCD$, and through P, a point on AC, the lines $GF \parallel AB$, $HE \parallel DA$, and the parts lettered as in the figure.

To prove that $b = b'$.

Proof. 1. $a + b + c = a' + b' + c'$,

$a = a', \quad c = c'$. I, prop. XXIV

2. $\therefore b = b'$. Ax. 3

149. Definitions. Since all rectangles which have two adjacent sides equal to two given lines a, b, are congruent (I, prop. XXVI, cor. 1), any such rectangle is spoken of as **the rectangle of *a* and *b*.**

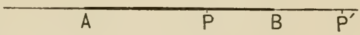
This is indicated by the symbol ab, or, if the adjacent sides are AB and CD, by the symbol $AB \cdot CD$. These symbols are read "The rectangle of a and b," "The rectangle of AB and CD," or, briefly, "The rectangle ab," "The rectangle AB (pause) CD." Since there is no multiplication of lines by lines, by any definition thus far known to the student, the readings "*a times b*," "*AB times CD*" are not recommended.

In like manner, any square whose side is equal to a given line is spoken of as **the square on** (or **of**) **that line.**

The square on a line AB is indicated by the symbol AB^2; on a line a by the symbol a^2; read "The square on (or of) AB," or, briefly, "AB-square"; and similarly for a.

Squares, rectangles, and polygons in general are often designated by the letters of two vertices not consecutive.

150. A point in a line-segment is said to **divide it internally**; a point in a produced part of a line-segment is said to **divide it externally**.



In the figure, AB is divided internally at P, and externally at P'. AP, PB are called *segments* of AB; and AP', $P'B$ are also called segments of AB.

The propriety of calling AP, PB, and AP', $P'B$, segments of AB is apparent, since $AP + PB = AB$, and also $AP' + P'B$ (which is negative) $= AB$.

Exercises. 185. If the sides BC, CA, AB, of $\triangle ABC$, are produced to X, Y, Z, respectively, so that $CX = BC$, $AY = CA$, $BZ = AB$, prove that $\triangle XYZ = 7 \cdot \triangle ABC$.

186. The medians of a triangle divide it into six equal triangles. (In what kind of a triangle are the six triangles congruent?)

187. Prove prop. III by bisecting BC at O, drawing DO to meet AB produced at D', and proving that $\triangle BD'O \cong \triangle CDO$, that $\triangle AD'D =$ trapezoid, etc.

188. Discuss prop. IV when P moves to C; through C on AC produced.

189. If two equal triangles are on opposite sides of a common base the line of that base bisects the line joining their vertices.

190. A triangle X is equal to a fixed triangle T and has a common base with T; what is the locus of the vertex of X? (Is the locus a single line or a pair of lines?)

191. P is any point on the diagonal BD of $\square ABCD$. Prove that $\triangle PAB = \triangle PBC$.

192. In ex. 191, suppose P moves to D; moves through D on BD produced.

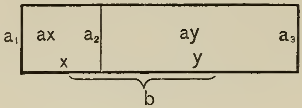
193. The sides AB, CA of a triangle are bisected in C', B', respectively; CC' cuts BB' at P. Prove that $\triangle PBC =$ quadrilateral $AC'PB'$.

194. If P is a point on the side AB, and Q a point on the opposite side CD of $\square ABCD$, prove that $\triangle PCD = \triangle QAB$.

195. If the mid-points of the sides of any convex quadrilateral are joined, in order, then (1) a parallelogram is formed, (2) which equals half the quadrilateral.

Proposition V.

151. Theorem. *The rectangle of two given lines equals the sum of the rectangles contained by one of them and the several segments into which the other is divided.*



Given the rectangle of a_1 and b, and b divided into the segments x and y.

To prove that $a_1b = a_1x + a_1y$.

Proof. 1. Let a_2 be drawn $\parallel a_1$ from the division point of b.

2. Then, in the figure, $a_1 = a_2 = a_3$. I, prop. XXIV

3. $\therefore a_1b = a_1x + a_1y$. Ax. 8

Corollaries. 1. *If a line is divided internally into two segments, the rectangle of the whole line and one segment equals the square on that segment together with the rectangle of the two segments.*

Make $a_1 = x$ in the proof above, and consider step 3.

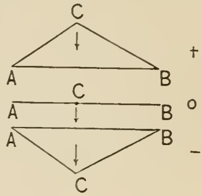
2. *If a line is divided internally into two segments, the square on the whole line equals the sum of the rectangles of the whole line and each of the segments.*

Make $a_1 = x + y$ in the proof above.

Note. This theorem is the geometric form of the **Distributive Law of Multiplication** of Algebra, which asserts that $a(x + y) = ax + ay$.

Exercise. 196. The rectangle of one line and the sum of two others equals the sum of the rectangles of the first and each of the other two. Consider the case where one of the second two lines is zero; negative.

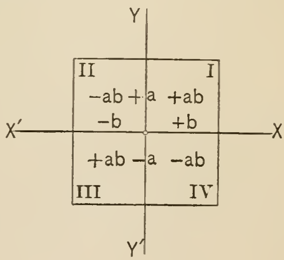
152. Positive and Negative Polygons. In general, a line AB is thought of as positive; but if, in the discussion of a proposition, A is thought of as approaching B, then, when A reaches B, AB becomes zero; and if A is thought of as passing through B, then AB is considered as having passed through zero and become negative; that is, $BA = -AB$.



A similar agreement exists as to triangles. In general, $\triangle ABC$ is thought of as positive; but if, in the discussion of a proposition, C moves down to rest on AB, then $\triangle ABC$ becomes zero; and as C passes through AB, $\triangle ABC$ passes through zero and is considered as having changed its sign and become negative; that is, $\triangle ACB = -\triangle ABC$.

In Book I, to accustom students to this convention (that $\triangle ACB = -\triangle ABC$), triangles were always named by taking the letters in the counter-clockwise (or positive) order, except in a few cases where a departure from this rule seemed advisable.

A similar agreement exists *as to rectangles*, which illustrates the law of signs in algebra. In the figure, I has for its altitude and base $+a$ and $+b$, and the rectangle is spoken of as $+ab$. But if b shrinks to zero, $+ab$ also shrinks to zero, and as b passes through zero and becomes negative, so ab is considered to pass through zero and to become negative; that is, II $= -ab$. If, now, a shrinks to zero, and passes through zero, changing its sign, so does $-ab$; that is, III $= +ab$. And finally, as $-b$ again passes through zero, so does ab, and therefore IV $= -ab$.



Exercise. 197. If P is any point in the plane of $\triangle ABC$, then $\triangle PAB + \triangle PBC + \triangle PCA = \triangle ABC$. (Monge.)

In the case of polygons in general, the law of signs will be readily understood from the annexed figures. In Figs. 1, 2, 3 both the upper and lower parts of the polygon are considered as positive; in Fig. 4, P

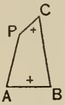


Fig. 1.

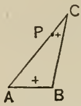


Fig. 2.

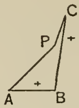


Fig. 3.

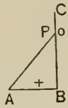


Fig. 4.

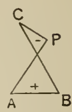


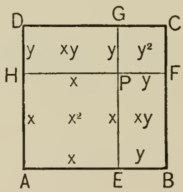
Fig. 5.

has reached BC and the upper part of the polygon has become zero; in Fig. 5, P has passed through BC and the upper part of the figure has passed through zero and become negative.

This treatment of negative surfaces dates from Meister (1769).

Proposition VI.

153. Theorem. *The square on the sum of two lines equals the sum of the squares on those lines plus twice their rectangle.*



Given $ABCD$, the square on $x + y$.

To prove that $(x + y)^2 = x^2 + y^2 + 2\,xy$.

Proof. 1. In the figure, let $AE = AH = x$, $EB = HD = y$, $EG \parallel AD$, $HF \parallel AB$, and HF cut EG at P.

2. Then the x's in the figure are all equal; also the y's. Def. □; I, prop. XXIV

3. $\therefore AP = x^2$, $PC = y^2$, § 99, def. □

and $EF = xy$, $HG = xy$. § 99, def. □

4. $\therefore (x + y)^2 = x^2 + y^2 + 2\,xy$. Ax. 8

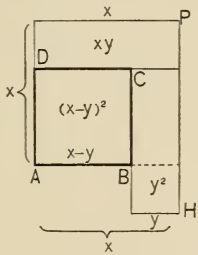
COROLLARIES. 1. *The square on a line equals four times the square on half that line.*

Make $x = y$ in step 4.

Then $$(2x)^2 = x^2 + x^2 + 2x^2,$$
or $$(2x)^2 = 4x^2.$$

That is, if $2x$ is the line, the square upon it equals four times the square on x.

2. *The square on the difference of two lines equals the sum of the squares on those lines minus twice their rectangle.*



For, in the above figure, $AP = x^2$, $BH = y^2$, $PD = xy$, $CH = xy$, and $AC = (x - y)^2$.

But $$AC = AP + BH - PD - CH,$$
or $$(x - y)^2 = x^2 + y^2 - 2xy.$$

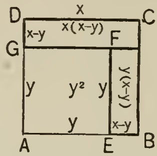
The truth of the corollary is, however, evident from prop. VI, if the agreement as to signs is considered; for if y becomes 0, then $2xy = 0$, and $y^2 = 0$; and as y passes through 0 and becomes negative, $2xy$ also becomes negative, but y^2 remains positive because it is the rectangle of $-y$ and $-y$.

Exercises. 198. If $ABC \ldots\ldots MNA$ is the perimeter of any polygon, and P is any point in the plane, then $\triangle PAB + \triangle PBC + \ldots\ldots + \triangle PMN + \triangle PNA$ is constant.

199. If A, B, C, D are four collinear points, in order, then $AB \cdot CD + AD \cdot BC = AC \cdot BD$. (Euler.) Investigate when B moves to and through C.

Proposition VII.

154. Theorem. *The difference of the squares on two lines equals the rectangle of the sum and difference of those lines.*



Given $ABCD$, a square on x, and $AEFG$, a square on y.

To prove that $x^2 - y^2 = (x + y)(x - y)$.

Proof. 1. Suppose the squares placed as in the figure, and GF produced to BC. Then the y's in the figure are all equal, as also the sides of x^2. Def. □

2. ∴ the $(x - y)$'s are equal. Ax. 3

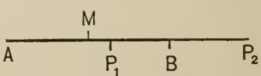
3. But $x^2 = y^2 + x(x - y) + y(x - y)$, Ax. 8

or $x^2 = y^2 + (x + y)(x - y)$. Prop. V

4. ∴ $x^2 - y^2 = (x + y)(x - y)$. Ax. 3

Corollary. *If a point divides a line internally or externally into two segments, the rectangle of the segments equals the difference of the square on half the line and the square on the segment between the mid-point of the line and the point of division.*

1. If AB is the line, P (either P_1 or P_2) the point of division, and M the mid-point, it is to be proved that



$$AP \cdot PB = AM^2 - MP^2.$$

2. Let $AB = y$, $AP = x$, then $PB = y - x$, $AM = \frac{1}{2}y$, and $MP = x - \frac{1}{2}y$.

3. But $x(y - x) = \left(\frac{y}{2}\right)^2 - \left(x - \frac{y}{2}\right)^2$, by prop. VII.

155. Reciprocity between Algebra and Geometry. From props. V, VI, VII, it is evident that a reciprocity exists between algebra and geometry which is likely to be of great advantage to each. This reciprocity will be more clearly seen by resorting to parallel columns.

Geometric Theorems.

If $x, y, \ldots\ldots$ are *line-segments*, and $xy, xz, \ldots\ldots$ represent the *rectangles* of x and y, x and $z, \ldots\ldots$, and $x(y+z)$ represents the *rectangle* of x and $y+z$, and x^2 represents the *square* on x, then

1. $x(y+z)=xy+xz.$ Prop. V
2. $(x+y)^2=x^2+y^2+2xy.$ Prop. VI
3. $x^2=4\left(\frac{x}{2}\right)^2.$ Prop. VI, cor. 1
4. $(x-y)^2=x^2+y^2-2xy.$ Prop. VI, cor. 2
5. $x^2-y^2=(x+y)(x-y).$ Prop. VII

Algebraic Theorems.

If $a, b, \ldots\ldots$ are *numbers*, and $ab, ac, \ldots\ldots$ represent the *products* of a and b, a and c, $\ldots\ldots$, and $a(b+c)$ represents the *product* of a and $b+c$, and a^2 represents the *second power* of a, then

1. $a(b+c)=ab+ac.$
2. $(a+b)^2=a^2+b^2+2ab.$
3. $a^2=4\left(\frac{a}{2}\right)^2.$
4. $(a-b)^2=a^2+b^2-2ab.$
5. $a^2-b^2=(a+b)(a-b).$

This correspondence of one symbol, one operation, one result, etc., of algebra, to one symbol, one operation, one result, etc., of geometry, or, as it is called, this "*one-to-one correspondence*," suggests many theorems of geometry that might otherwise remain unnoticed. This correspondence is the basis of the treatment of Proportion, Book IV.

Exercise. 200. Prove geometrically that $(x+y)^2-(x-y)^2=4xy$.

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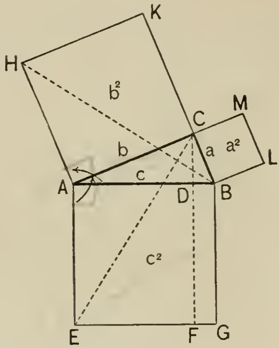
Proposition VIII.

156. Theorem. *In a right-angled triangle the square on the hypotenuse equals the sum of the squares on the other two sides.*



Given the right-angled $\triangle ABC$, $\angle C$ being right.

To prove that $a^2 + b^2 = c^2$.

Proof. 1. Let $BLMC = a^2$, $ACKH = b^2$, $AEGB = c^2$; suppose $CF \parallel AE$, and HB and CE drawn.

2. Then $\because$ $\angle$s KCA, ACB, are right, their sum = st. $\angle$, and $\therefore$ BCK is a st. line. § 14, def. st. $\angle$

3. And $\because$ $\angle CAH = \angle EAB$, Prel. prop. I
$\therefore$ $\angle BAH = \angle EAC$, by adding $\angle BAC$. Ax. 2

4. And $\because$ $AC = HA$, and $AE = BA$, § 99, def. $\square$
$\therefore$ $\triangle ABH \cong \triangle AEC$. Why?

5. But $\square AF =$ twice $\triangle AEC$,
and $b^2 =$ twice $\triangle ABH$. Why?
$\therefore$ $\square AF$, or $AB \cdot AD$, $= b^2$. Ax. 6

6. Similarly, $\square BF$, or $BA \cdot BD$, $= a^2$.

7. $\therefore$ $a^2 + b^2 = \square AF + \square BF = c^2$. Axs. 2, 8

157. NOTE. The first proof of this theorem is said to have been given by Pythagoras about 540 B.C., although the theorem itself was known long before that time. From this fact it is generally known as the **Pythagorean Proposition.** It is one of the most important in geometry.

There have been many proofs devised for the Pythagorean proposition. In the subsequent exercises occasional proofs will be suggested, that the student may see the great variety of ways in which the theorem may be attacked. That the proposition would naturally be suggested to a people using tile floors is seen from Fig. 1, although the proof following

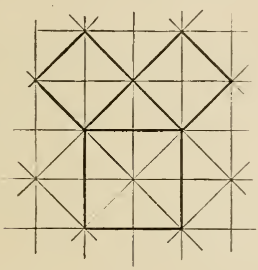


FIG. 1.

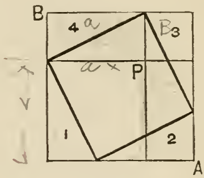


FIG. 2.

from such a figure is special, being limited to the case of the isosceles right-angled triangle.

In Fig. 2 is given a suggestion of the conjectured proof of Pythagoras: If △ 1, 2, 3, 4 are taken from the figure, the square on the hypotenuse remains; and if the two ▭ *AP*, *PB*, are taken away, the sum of the squares on the two sides remains; but since the two rectangles equal the four triangles, these remainders are equal.

Exercises. 201. What is the use of steps 2 and 3 in the proof of prop. VIII?

202. In the figure of prop. VIII, prove that $AK \parallel BM$.

203. Also that H, C, L are collinear.

204. Twice the sum of the squares on the medians of a right-angled triangle equals thrice the square on the hypotenuse.

Fig. 3 is that of Bhaskara, the Hindu: The inside square is evidently $(a-b)^2$, and each of the four triangles is $\frac{1}{2}ab$; $\therefore c^2 - 4 \cdot \frac{1}{2}ab = (a-b)^2 = a^2 + b^2 - 2ab$; $\therefore c^2 = a^2 + b^2$.

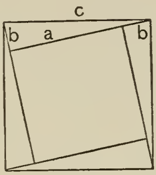


FIG. 3.

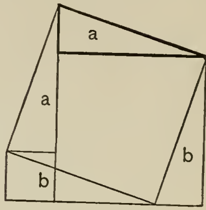


FIG. 4.

Fig. 4 is one of the most simple: If from the whole figure there are taken $\triangle$ b, there remains the square on the hypotenuse; or if the equal $\triangle$ a are taken, there remains the sum of the squares on the two sides.

158. Definition. The **projection of a point on a line** is the foot of the perpendicular from the point to the line.

Thus A' and B', Figs. 1, 2, are the projections of A and B on $X'X$.

The **projection of a line-segment** on another line in the same plane is the segment cut off by the projections of its end-points, *e.g.* in Figs. 1 and 2, $A'B'$ is the projection of AB.

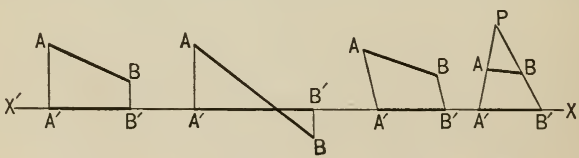
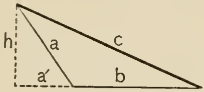


FIG. 1. FIG. 2. FIG. 3. FIG. 4.

Strictly these are *orthogonal* (or right-angled) projections; but since orthogonal projections are the only kind ordinarily considered in elementary geometry, they are called, simply, *projections*. In advanced geometry, the projections of Figs. 3, 4 are among the others used. Fig. 3 represents an oblique projection, and Fig. 4 represents a projection from a point. Fig. 4 is the most general, approaching the others as P recedes to a greater distance.

PROPOSITION IX.

159. Theorem. *In an obtuse-angled triangle the square on the side opposite the obtuse angle equals the sum of the squares on the other two sides, together with twice the rectangle of either side and the projection of the other on the line of that side.*



Given $\triangle\, abc$, obtuse-angled opposite c, and a' the projection of a on the line of b.

To prove that $c^2 = a^2 + b^2 + 2\,ba'$.

Proof. 1. In the figure, $h \perp a'$, § 158, def. projection

$\therefore h^2 + (a' + b)^2 = c^2,$ Why?

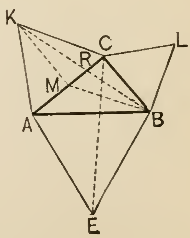
or $h^2 + a'^2 + b^2 + 2\,a'b = c^2.$ Prop. VI

2. $\therefore a^2 + b^2 + 2\,a'b = c^2.$ Prop. VIII

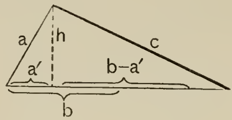
Exercises. 205. In the figure of prop. VIII, prove that, if HK and LM are produced to meet at P, then $AE =$ and $\parallel PC$, and $BG =$ and $\parallel PC$.

206. If the diagonals of a quadrilateral intersect at right angles, prove that the sum of the squares on one pair of opposite sides equals the sum of the squares on the other pair.

207. In the annexed figure, equilateral triangles are constructed on the sides of a right-angled triangle; M is the mid-point of CA. Prove (1) $\triangle\, ABK \cong \triangle\, AEC$, (2) $MK \parallel BC$, (3) $\triangle\, BCM = \triangle\, BCK$, (4) $\triangle\, BRM = \triangle\, RCK$, (5) $\triangle\, ABK = \triangle\, ACK + \triangle\, ABM = \triangle\, ACK + \frac{1}{2}\triangle\, ABC$, (6) $\therefore$ from (1) and (5) $\triangle\, AEC = \triangle\, ACK + \frac{1}{2}\triangle\, ABC$, (7) similarly, $\triangle\, CEB = \triangle\, BLC + \frac{1}{2}\triangle\, ABC$, (8) $\therefore$ figure $AEBC =$ figure $ABLCK$, (9) $\therefore \triangle\, AEB = \triangle\, BLC + \triangle\, ACK$. State in full form the theorem proved in (9).



Corollaries. 1. *In any triangle the square on the side opposite an acute angle equals the sum of the squares on the other two sides, less twice the rectangle of either side and the projection of the other side on it.*



For, in the above figure,

$$h^2 + (b - a')^2 = c^2.$$

$$\therefore h^2 + b^2 + a'^2 - 2\,a'b = c^2.$$

$$\therefore a^2 + b^2 - 2\,a'b = c^2.$$

The truth of the corollary is, however, evident from prop. IX; for if $\angle ba$ becomes 90°, $a' = 0$ and prop. IX becomes prop. VIII; and if $\angle ba$ becomes acute, a' passes through 0 and becomes negative, and $\square\, a'b$ becomes negative; $\therefore$ step 2 becomes $a^2 + b^2 - 2\,a'b = c^2$.

2. *Converse of props. VIII, IX, and prop. IX, cor. 1. The angle opposite a given side of a triangle is right, obtuse, or acute, according as the square on that side is equal to, greater or less than the sum of the squares on the other two sides.*

Law of Converse (§ 73). Write out the proof in full.

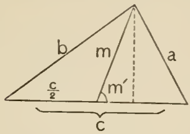
Exercises. **208.** In the figure of prop. VIII, the medians of $\triangle ABC$ are perpendicular to and equal to half of KM, HE, LG, respectively. (Complete the $\square\, BCAV$, and prove $CV =$ and $\perp KM$, etc.)

209. XOY is any angle, and from B, on OY, BA is drawn $\perp OX$; from B is drawn $BZ \parallel OX$; now *if* P can be found on BA, so that OP produced to cut BZ in Q, makes $PQ = 2\,OB$, then $\angle XOQ = \frac{1}{3} \angle XOY$. (That is, $\angle XOY$ is trisected. It has been *proved* that this famous problem of the Greeks, to trisect any angle, cannot be solved by elementary geometry, that is, by using the compasses and straight-edge only. There are various solutions if other instruments are allowed.)

210. Prove algebraically that if n is an even number, then n, $\frac{1}{4} n^2 - 1$, $\frac{1}{4} n^2 + 1$ are numerically the sides of a right-angled triangle (Plato), and that they are integers.

PROPOSITION X.

160. Theorem. *The sum of the squares on any two sides of a triangle equals twice the sum of the squares on one-half the third side and on the median to that side.*



Given the $\triangle abc$, and m the median to c.

To prove that $a^2 + b^2 = 2\left[\left(\frac{c}{2}\right)^2 + m^2\right]$.

Proof. 1. Let m' be the projection of m on c, and suppose $\angle cm$ acute.

2. Then $a^2 = \left(\frac{c}{2}\right)^2 + m^2 - 2\left(\frac{c}{2}\right)m'$, Prop. IX, cor. 1

and $b^2 = \left(\frac{c}{2}\right)^2 + m^2 + 2\left(\frac{c}{2}\right)m'$. Why?

3. $\therefore a^2 + b^2 = 2\left[\left(\frac{c}{2}\right)^2 + m^2\right]$. Ax. 2

If $\angle cm$ is obtuse, then $\angle mc$ is acute, and the proof merely interchanges a, b without affecting step 3. If $\angle cm$ is right, then $m' = 0$ in step 2, but 3 is not affected.

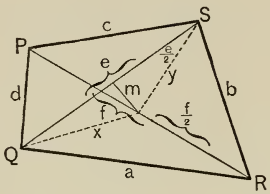
Exercises. 211. In prop. X, prove that $4m^2 = 2(a^2 + b^2) - c^2$. Hence show that in a right-angled triangle (in which $a^2 + b^2 = c^2$) the median to the hypotenuse equals half the hypotenuse.

212. From ex. 211, what is the locus of the vertex of the right angle of a right-angled triangle with a given hypotenuse?

213. The sides of a triangle are 10, 12, 15 inches. Is the triangle right-angled? obtuse-angled?

Proposition XI.

161. Theorem. *The sum of the squares on the sides of a quadrilateral equals the sum of the squares on the diagonals plus four times the square on the line joining the mid-points of the diagonals.*



Given a quadrilateral *abcd*, convex, concave, or cross, with diagonals e, f, and with m joining the mid-points of e, f.

To prove that $a^2 + b^2 + c^2 + d^2 = e^2 + f^2 + 4\,m^2$.

Proof. 1. In the figure, $a^2 + d^2 = 2\,x^2 + 2\left(\frac{f}{2}\right)^2$, Why?

and $b^2 + c^2 = 2\,y^2 + 2\left(\frac{f}{2}\right)^2$. Why?

2. $\therefore\ a^2 + b^2 + c^2 + d^2 = 2\,(x^2 + y^2) + 4\left(\frac{f}{2}\right)^2$ Ax. 2

$= 2\left[2\left(\frac{e}{2}\right)^2 + 2\,m^2\right] + 4\left(\frac{f}{2}\right)^2$. Prop. X

$= e^2 + f^2 + 4\,m^2$. Prop. VI, cor. 1

Corollary. *The sum of the squares on the diagonals of a parallelogram equals the sum of the squares on the sides.*

For then $m = 0$; I, prop. XXIV, cor. 2.

Note. The theorem is due to Euler. The corollary was, however, known to the Greeks.

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2. PROBLEMS.

PROPOSITION XII.

162. Problem. *To construct a triangle equal to a given polygon.*

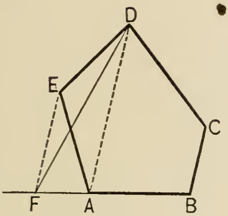


FIG. 1.

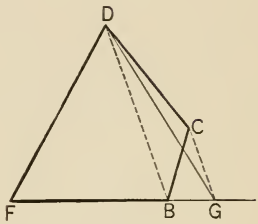


FIG. 2.

Given polygon $ABCDE$.

Required to construct a $\triangle$ equal to $ABCDE$.

Construction. 1. Produce BA, join D and A, draw $EF \parallel DA$, meeting BA produced at F; draw DF.

§ 28, post. of st. line; I, prop. XXXIII

2. Then polygon $FBCD$ has one less side than $ABCDE$, and will be proved equal to it. Continue the process until a $\triangle$ is reached (Fig. 2).

Proof. 1. $\because EF \parallel DA$,

$\therefore \triangle ADF = \triangle ADE$, having same base AD.

Prop. II

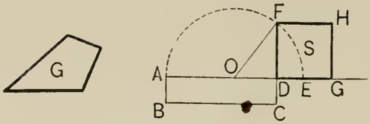
2. Adding polygon $ABCD$, $FBCD = ABCDE$. Ax. 2

3. Similarly thereafter. In Fig. 2, $\triangle FGD$ is the triangle required.

Exercise. 214. To construct a rhombus equal to a given parallelogram, and on the same base. Discuss for impossible cases.

Proposition XIII.

163. Problem. *To construct a square equal to a given polygon.*



Given polygon G.

Required to construct a square equal to G.

Construction. 1. Construct a $\triangle$ equal to G. Prop. XII

2. By drawing a line through the vertex of this $\triangle \parallel$ to the base, and erecting $\perp$'s from an extremity and the mid-point of the base, construct a $\square$, as $ABCD$, equal to this $\triangle$. I, props. XXIX, XXXI, XXXIII

Then if $AB = DA$, $ABCD$ is the required $\square$.

3. If not, produce AD to E, making $DE = CD$; § 28

bisect AE at O, I, prop. XXXI

and with center O and radius OE, describe a semi-circumference. § 109, post. of $\odot$

4. Produce CD to meet circumference at F, § 28

and construct a square on DF. I, prop. XXXIX

Then DF^2, S in the figure, is the required $\square$.

Proof. 1. Draw OF, let $r = OF = OA = OE$, and $x = OD$;

then $CD = DE = r - x$,

and $AD = r + x$.

2. Then $(r + x)(r - x) = r^2 - x^2$ Prop. VII

$= x^2 + DF^2 - x^2 = DF^2$. Why?

3. But $(r + x)(r - x) = \square ABCD = G$, Const. 2

and $\therefore DF^2 =$ polygon G. Ax. 1

EXERCISES.

215. If one angle of a triangle is two-thirds of a straight angle, show that the square on the opposite side equals the sum of the squares on the other two sides, together with their rectangle.

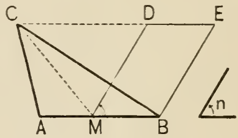
216. Prove prop. XI for a concave quadrilateral.

217. If $\angle P = 180°$ and $SP = PQ$, show that prop. XI reduces to a previous theorem.

218. Prove prop. XI, cor. directly from prop. X without reference to prop. XI.

219. If $ABCD$ is any quadrilateral, and the mid-points of the diagonals are joined by a line bisected at M, and if P is any point, then $PA^2 + PB^2 + PC^2 + PD^2 = MA^2 + MB^2 + MC^2 + MD^2 + 4\,PM^2$.

220. To construct a parallelogram equal to a given triangle, and having one of its angles equal to a given angle.



221. To construct a parallelogram equal to a given square, on the same base and having an angle equal to half the angle of the square.

222. To construct an isosceles triangle equal to a given triangle, and on the same base.

223. To construct a triangle equal to a given parallelogram, and having one of its angles equal to a given angle.

224. To construct a parallelogram equal to a given triangle, and having its perimeter equal to that of the triangle. (In the figure of ex. 220 how must MD compare with $BC + CA$?)

225. To construct a square equal to the sum of two given squares. (Apply prop. VIII.)

226. On a given line to construct a rectangle equal to a given rectangle.

227. On one side of a triangle as a diagonal to construct a rhombus equal to the given triangle.

228. Prove that in any triangle three times the sum of the squares on the sides equals four times the sum of the squares on the three medians.

229. Also that three times the sum of the squares on the lines joining the centroid to the vertices equals the sum of the squares on the sides.

230. If one angle of a triangle is one-third of a straight angle, show that the square on the opposite side equals the sum of the squares on the other two sides less their rectangle.

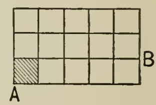
3. PRACTICAL MENSURATION.

164. For practical purposes a surface is measured as follows:

1. A square unit is defined as a square which is one linear unit long and one linear unit wide.

That is, a square inch is a square that is 1 in. long and 1 in. wide; a square meter is a square that is 1 m. long and 1 m. wide, etc. In the figure the shaded square is considered as a square unit.

2. If two sides of a rectangle are 3 in. and 5 in. respectively, then, in the figure, the area of the strip AB is 5×1 sq. in., and the total area is $3 \times 5 \times 1$ sq. in., or 15 sq. in.



Theoretically, a rectangle rarely has sides both of which exactly contain any linear unit, however small. Such cases are discussed in Book IV. But for practical purposes the above method is approximate to any required degree.

At present it is necessary for the student to learn that geometry gives him an instrument for practical work. It will accordingly be assumed that the measurements can be made to any degree of approximation, and that the expressions *area*, *measure*, etc., are understood in their ordinary sense. It has already been explained that the rectangle of two lines corresponds to the product of two numbers; hence, in practice, lines are represented by numbers, and their rectangles by the products of those numbers. This practical measurement will be exemplified hereafter, as it has already been to some extent, in the numerical exercises.

Exercises. 231. A field is in the form of a rhombus, the obtuse angle being twice the acute angle; the shorter diagonal is 300 feet. Find the area of the field in square feet.

232. A railroad embankment extends through a farm 1 mile long, its rails being in straight lines perpendicular to the two parallel sides of the farm; the embankment is 80 ft. wide at the bottom at one end, and 60 ft. at the other. How much land was taken for railroad purposes?

EXERCISES.

233. A road running across a farm is $\frac{1}{4}$ mile long and 3 rods wide; the road being rectangular, find its area in acres.

234. The side of an equilateral triangle is 15. Find the area.

235. In excavating for a canal 30 ft. deep, 200 ft. wide at the top, and 160 ft. wide at the bottom, what is the area of a cross-section?

236. One diagonal of a quadrilateral is 100, and the perpendiculars, from the other two vertices, upon it, are 50 and 40. Find the area.

237. The area of a triangle is a and the altitude is h. Find the base. Investigate for $a = 325.85$, $h = 38$; also for $a = 100$, $h = 100$.

238. The area of a trapezoid is a and the two bases are b_1, b_2. Find the altitude. Investigate for $a = 223.375$, $b_1 = 13.5$, $b_2 = 6.4$; also for $a = 10$, $b_1 = 0$, $b_2 = 10$.

239. The area of a trapezoid is 542.5, the altitude is 21.7, and the difference between the parallel sides is 11.2. Find those sides.

240. The area of a square is 2. Find the side of a square of twice the area; thrice the area; four times the area.

241. The altitude of an equilateral triangle is 160. Find the area.

242. The base of an isosceles triangle is $\frac{8}{5}$ of one of the equal sides, and the altitude is 10. Find the area.

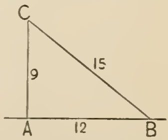
243. Two sides of a right-angled triangle are 1036 and 1173. Find the hypotenuse and the area.

244. Find to three decimal places the diagonal of a square whose area is 1.

245. In a right-angled triangle the perpendicular from the vertex of the right angle divides the hypotenuse into two segments, 2.88 and 5.12. Find the two sides.

246. From the vertex A of $\triangle ABC$, $AD \perp BC$. Find the lengths of BD, CD, knowing that $AB = 307.8$, $CA = 480.168$, $BC = 689.472$.

247. A surveyor, wishing to erect a perpendicular to a line on the ground, drives two stakes, A, B, 12 links apart; to these he fastens the ends of a 24-link segment, and stretches the chain, at the end of the 9th link from A, to C. Show that $AC \perp AB$. (This method of erecting perpendiculars was known to the temple and pyramid builders, and surveyors employed for this purpose were called "rope stretchers." The method is still used in practical field work.)



BOOK III.—CIRCLES.

165. Definitions. A **circle** is the finite portion of a plane bounded by a curve, which is called the **circumference**, and is such that all points on that line are equidistant from a point within the figure called the **center** of the circle.

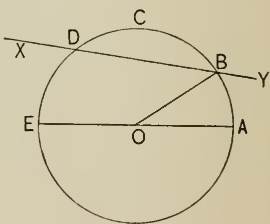
For corollaries and postulates, see §§ 108, 109.

Certain definitions are here repeated for convenience.

If two equal figures are necessarily congruent, as in the case of circles, angles, squares, and line-segments, the word *equal* is ordinarily used to express congruence. Hence congruent circles (see § 108, 2) are ordinarily called simply *equal*.

A straight line terminated by the center and the circumference is called a **radius**.

A straight line through the center terminated both ways by the circumference is called a **diameter**.



166. The straight line joining any two points on a circumference is called a **chord**.

Hence a diameter is a chord passing through the center. In the figure, AE and BD are chords.

The expressions center, radius, diameter, chord, of a *circumference* are sometimes used instead of center, etc., of a *circle*.

167. The line of which a chord is a segment is called a **secant**, as XY in the figure.

168. A part of a circumference is called an **arc**.

In the figure, BCD is an arc. As in naming an angle, the counter-clockwise order is followed, and arcs so named are considered positive.

169. One-half of a circumference is called a **semicircumference.**

170. A fourth part of a circumference is called a **quadrant.**

171. An angle formed by two radii is called a **central angle.**

In the figure, $\angle s\ AOB$, BOE are central angles.

172. A central angle is said to **stand upon** the arc which lies within the angle and is cut off by the arms.

$\angle s\ AOB$, BOE stand upon $\overset{\frown}{AB}$, $\overset{\frown}{BE}$, respectively.

173. The arc upon which the sum of two central angles stands is called the **sum of the arcs** upon which those angles stand. Similarly for the **difference of two arcs.**

Thus, $\overset{\frown}{AE} = \overset{\frown}{AB} + \overset{\frown}{BE}$, and $\overset{\frown}{AB} = \overset{\frown}{AD} - \overset{\frown}{BD}$.

174. Two arcs are said to be **complements** of each other if their sum is a quadrant; **supplements** of each other if their sum is a semicircumference; **conjugates** of each other if their sum is a circumference.

In the figure, $\overset{\frown}{AB}$ is the supplement of $\overset{\frown}{BE}$ and the conjugate of $\overset{\frown}{BA}$.

175. An arc greater than a semicircumference is called a **major arc**; one less than a semicircumference, a **minor arc.**

In the figure, AB, BD, DE are minor arcs; DEA is a major arc.

176. Conjugate arcs are said to be **subtended** by their common chord.

In the figure, $\overset{\frown}{BD}$ and $\overset{\frown}{DB}$ are each said to be *subtended* by chord BD. The word *subtend* is variously used in geometry. It means to *extend under* or to *be opposite to.* Hence in a triangle a side is said to subtend an opposite angle, a chord is said to subtend an arc, etc.

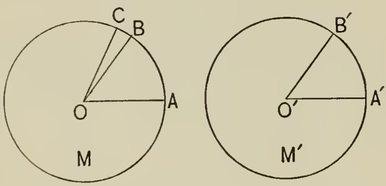
177. A portion of a circle cut off by an arc and two radii drawn to its extremities is called a **sector,** and the central angle standing on that arc is called the **angle of the sector.**

In the figure, OAB is a sector, and $\angle AOB$ is its angle.

1. CENTRAL ANGLES.

PROPOSITION I.

178. Theorem. *In the same circle or in equal circles, if two central angles are equal, the arcs on which they stand are equal also, and of two unequal central angles the greater stands on the greater arc.*



Given M, M', two equal circles, and central angles

$$AOB = A'O'B', \quad AOC > A'O'B'.$$

To prove that $\widehat{AB} = \widehat{A'B'}$, and $\widehat{AC} > \widehat{A'B'}$.

Proof. 1. Place $\odot\, M'$ on $\odot\, M$ so that $\angle\, A'O'B'$ coincides with its equal $\angle\, AOB$.

Then A' coincides with A, and B' with B.

§ 165, def. $\odot$

2. Then $\widehat{A'B'}$ coincides with $\widehat{AB}$, because its points are equidistant from O. § 165, def. $\odot$

3. Also, $\because \angle\, AOC > \angle\, A'O'B'$,

$\therefore \angle\, AOC > \angle\, AOB$.

4. $\therefore C$ is not in $\angle\, AOB$, and $\widehat{AC} > \widehat{AB}$. Ax. 8

5. And $\because \widehat{AB} = \widehat{A'B'}$, $\therefore \widehat{AC} > \widehat{A'B'}$. Ax. 9

The proof is essentially the same for a single circle, and so in general when equal circles are involved.

COROLLARIES. 1. *Sectors of the same circle or of equal circles, which have equal angles, are equal.*

For, by steps 1, 2, they coincide.

2. *Sectors of the same circle, or of equal circles, which have unequal angles, are unequal, the greater having the greater angle.*

This is proved by superposition in steps 3. 4. 5, of the proposition.

3. *The two arcs into which the circumference is divided by a diameter are equal.*

For their central angles are straight angles, and these being equal the arcs are equal by the proposition.

4. *The two figures into which a circle is divided by a diameter are equal.*

For their central angles are straight angles. Hence by cor. 1 they are equal.

This corollary is attributed to Thales.

179. Definition. The figure formed by a semicircumference and the diameter joining its extremities is called a **semicircle.**

It is proved (cor. 4) that all semicircles, cut from the same circle, are equal. Hence the name, *semi-* meaning half.

180. Since the 360 equal angles, into which the perigon at the center of a circle is imagined to be divided, stand on equal arcs by prop. I. the ordinary mensuration of angles by degrees is also used for arcs. Similarly for minutes, seconds, and other measurements. Hence the common expression, *an angle at the center is measured by the subtended arc.*

The expression is not strictly correct; we do not measure an angle by an arc, but the angle and arc have the same *numerical* measure, as will be proved in § 254. We might as truly say that an arc is measured by its central angle. But the expression is so commonly used, and has found its way into so many text-books and examination papers, that the student needs to become familiar with it.

Proposition II.

181. Theorem. *In the same circle or in equal circles, if two arcs are equal, the central angles which they subtend are equal also, and of two unequal arcs the greater subtends the greater central angle.*

Proof. If O and O' are two central angles, and A, A' are the arcs on which they stand, it has been proved in prop. I that

$$\begin{array}{llll} \text{If} & O > O', & \text{then} & A > A', \\ \text{“} & O = O', & \text{“} & A = A', \\ \text{“} & O < O', & \text{“} & A < A'. \end{array}$$

Hence the converses are true, by the Law of Converse, § 73.

Corollaries. 1. *In the same circle or in equal circles, equal sectors have equal angles; and of two unequal sectors, the greater has the greater angle.*

Law of Converse, § 73, from prop. I, cors. 1, 2.

2. *A central angle is greater than, equal to, or less than, a right angle, according as the arc on which it stands is greater than, equal to, or less than, a quadrant.* (Why?)

Exercises. 248. If two lines drawn to a circumference, from a point within the circle, are equal, they subtend equal central angles.

249. Prove the converse of ex. 248.

250. Two circumferences cannot bisect each other.

251. Suppose from the point P on a circumference two equal chords, PA, PB, are drawn. Prove (1) that these chords subtend equal central angles, (2) that they subtend equal arcs.

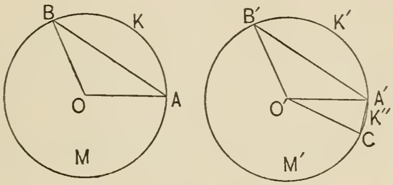
252. The arc AB is bisected by the point M, and MC is a diameter; prove that chord AC = chord BC.

253. How many degrees in the central angle standing on a third of a circumference? a fourth? a fifth? .....

2. CHORDS AND TANGENTS.

PROPOSITION III.

182. Theorem. *In the same circle or in equal circles, if two arcs are equal they are subtended by equal chords, and of two unequal minor arcs the greater is subtended by the greater chord.*



Given two equal circles, M, M'; two equal arcs, K, K'; and two unequal minor arcs, $K > K''$.

To prove that, as lettered in the figure, chords $AB = A'B'$, $AB > CA'$.

Proof. 1. Draw the radii OA, OB, $O'A'$, $O'B'$, $O'C$. Then

$\because \widehat{K} = \widehat{K'}$, $\therefore \angle AOB = \angle A'O'B'$. Prop. II

2. But $\because \odot M = \odot M'$,

$\therefore OA = OB = O'A' = O'B' = O'C$.

3. $\therefore \triangle OAB \cong \triangle O'A'B'$, and $AB = A'B'$. Why?

4. Also,

$\because \widehat{K} > \widehat{K''}$, $\therefore \angle AOB > \angle CO'A'$, Prop. II

$\therefore AB > CA'$. I, prop. X

COROLLARY. *In the same circle or in equal circles, of two unequal major arcs, the greater is subtended by the less chord.*

Proposition IV.

183. Theorem. *In the same circle or in equal circles, if two chords are equal they subtend equal major and equal minor arcs; and of two unequal chords the greater subtends the greater minor and the less major arc.*

Proof. Let C, C' be two chords of the same circle or of equal circles; N, N' their corresponding minor arcs;
J, J' " " major arcs.

From prop. III,

if $N > N'$, or if $J < J'$, then $C > C'$,
" $N = N'$, " " $J = J'$, " $C = C'$,
" $N < N'$, " " $J > J'$, " $C < C'$.

Hence the converses are true, by the Law of Converse, § 73.

Exercises. 254. If through a point in a circle two chords are drawn making equal angles with the diameter through that point, these chords cut off equal arcs of the circle.

255. The intersecting chords joining the extremities of two equal arcs of a circle are equal.

256. What is meant by an arc of 75°? by one of 300°? Can the sum of two arcs ever exceed an arc of 360°? Draw a figure to illustrate your answer.

257. Does the chord subtending the arc $2a$ equal twice the chord subtending the arc a? Prove your statement.

258. May the chord subtending the arc $2a$ ever equal the chord subtending the arc a? If not, show why; if so, tell how many degrees in the arc a.

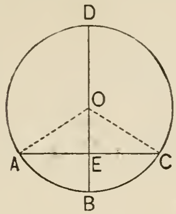
259. How many degrees in the supplement of the arc 90°? 175°? 180°? 190°?

260. How many degrees in the conjugate of the arc 180°? 300°? 360°? 400°?

261. How does the length of the chord subtending an arc of 60° compare with that subtending an arc of 90°? 300°? (Call the radius r, and determine each in terms of r.)

PROPOSITION V.

184. Theorem. *A diameter which is perpendicular to a chord bisects the chord and its subtended arcs.*



Given the diameter BD perpendicular to chord AC at E.

To prove that (1) $AE = EC$,

(2) $\widehat{AB} = \widehat{BC}$, (3) $\widehat{DA} = \widehat{CD}$.

Proof. 1. Drawing radii OA, OC, then

$OA = OC$, § 109, post. of $\odot$

and $AE = EC$, I, prop. XX, cor. 6

$\therefore \angle AOE = \angle EOC$. I, prop. XX, cor. 5

2. $\therefore \widehat{AB} = \widehat{BC}$. Why?

3. And $\because \angle DOA = \angle COD$, Prel. prop. IV

$\therefore \widehat{DA} = \widehat{CD}$. Why?

COROLLARIES. 1. *Conversely, a diameter which bisects a chord is perpendicular to it.*

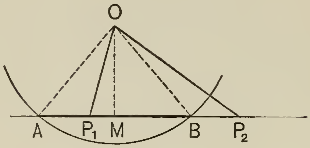
For $\because AE = EC$, and $OA = OC$, $\therefore DB$ has two points equidistant from A and C. Hence, being determined by these points, it is $\perp$ to AC, by I, prop. XLI.

2. *The perpendicular bisector of a chord passes through the center of the circle and bisects the subtended arcs.*

For the center is equidistant from the ends of the chord, by definition of a circle; $\therefore$ it lies on the perpendicular bisector of the chord, by I. prop. XLI.

Proposition VI.

185. Theorem. *All points in a chord lie within the circle; and all points in the same line, but not in the chord, lie without the circle.*



Given the points P_1 in a chord AB, and P_2 in AB produced.

To prove that P_1 is within the circle, and P_2 is without.

Proof. 1. Suppose O the center, and OA, OB, OP_1, OP_2 drawn, and $OM \perp AB$.

Then M is between A and B. Prop. V

2. And $\because \angle AOM > \angle P_1OM$,

$\therefore AO > P_1O$, I, prop. XX

and $\therefore P_1$ is within the $\odot$. § 108, def. $\odot$, cor. 3

3. And $\because \angle MOP_2 > \angle MOB$,

$\therefore P_2O > BO$, I, prop. XX

and $\therefore P_2$ is without the $\odot$. § 108, def. $\odot$, cor. 3

Corollary. *A straight line cannot meet a circumference in more than two points.*

For every other point on that line must be either between or not between those two points, and hence must lie either within or without the circle.

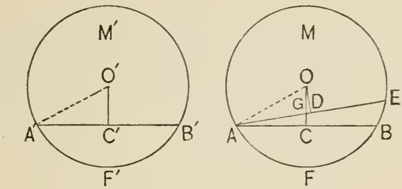
---

6 **Exercises.** 262. Prove that, in general, two chords of a circle cannot bisect each other. What is the exception?

263. What is the locus of the mid-points of a pencil of parallel chords of a circle? Why?

PROPOSITION VII.

186. Theorem. *In the same circle or in equal circles, equal chords are equidistant from the center; and of two unequal chords the greater is nearer the center.*



Given two equal ⓢ M, M', with chords $AB = A'B'$, $AE > A'B'$, and OC, OD, $O'C'$ $\perp$'s from center O to AB, AE, and from center O' to $A'B'$.

To prove that (1) $OC = O'C'$, (2) $OD < O'C'$

Proof. 1. C, C' bisect AB, $A'B'$, Prop. V

$\therefore AC = A'C'$, being halves of equal chords. Ax. 7

2. Draw OA, $O'A'$; then $\because OA = O'A'$,

and $\angle C = \angle C'$, Prel. prop. I

$\therefore \triangle ACO \cong \triangle A'C'O'$, I, prop. XIX, cor. 5

and $OC = O'C'$, which proves (1).

3. And $\because AE > A'B'$,

then $AE > AB$, which equals $A'B'$. Ax. 9

4. $\therefore$ minor $\overset{\frown}{AFE} > \overset{\frown}{AFB}$,

so that E does not lie on $\overset{\frown}{AFB}$. Prop. IV

5. And $\because O$ and AB are on opposite sides of AE,

$\therefore OC$ cuts AE, as at G, and $OD < OG$.

I. prop. XX

6. And $\because OG < OC$, $\therefore OD < OC$. Ax. 9

7. And $\because OC = O'C'$, $\therefore OD < O'C'$. Ax. 9

PROPOSITION VIII.

187. Theorem. *In the same circle or in equal circles, chords that are equidistant from the center are equal; and of two chords unequally distant, the one nearer the center is the greater.*

Proof. If c, c' are two chords of the same circle or of equal circles, and d, d' are the respective perpendiculars from the center upon them; then from prop. VII,

$$\begin{array}{llll} \text{If} & c > c', & \text{then} & d < d', \\ `` & c = c', & `` & d = d', \\ `` & c < c', & `` & d > d'. \end{array}$$

Hence the converses are true by the Law of Converse, § 73.

COROLLARY. *The diameter is the greatest chord in a circle.* For its distance from the center is zero.

Exercises. 264. AB is a fixed chord of a circle, and XY is any other chord having its mid-point P on AB. What is the greatest and what is the least length that XY can have?

265. What is the locus of the mid-points of equal chords of a circle?

266. Two parallel chords of a circle are 6 inches and 8 inches, respectively, and the distance between them is 1 inch. Find the radius.

267. Two chords are drawn through a point on a circumference so as to make equal angles with the radius drawn to that point. Prove that the chords subtend equal arcs.

268. If from the extremities of any diameter perpendiculars to any secant are drawn, the segments between the feet of the perpendiculars and the circumference will be equal. Draw the various figures.

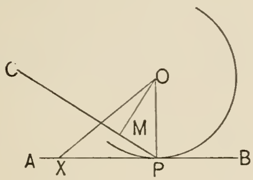
269. If two equal chords of a circle intersect, the segments of the one are equal respectively to the segments of the other.

270. Find the shortest chord which can be drawn through a given point in a circle.

271. The circumference of a circle whose center lies on the bisector of an angle cuts equal chords, if any, from the arms.

PROPOSITION IX.

188. Theorem. *Of all lines passing through a point on a circumference, the perpendicular to the radius drawn to that point is the only one that does not meet the circumference again.*



Given point P on the circumference of a ⊙ with center O, and AB, PC, respectively perpendicular and oblique to OP at P.

To prove that AB does not meet the circumference again, but that PC does.

Proof 1. Let $OM \perp PC$, and OX be any oblique to AB.

Then $OM < OP$, I, prop. XX

and $\therefore M$ is within the ⊙, and PC cuts the circumference again. §§ 108, 109

2. Also, $OX > OP$, Why?

and $\therefore X$, any point except P on AB, is without the ⊙. § 108, def. ⊙, cor. 3

3. $\therefore$ the perpendicular does not meet the circumference again, but an oblique does.

189. Definitions. The unlimited straight line which meets the circumference of a circle in but one point is said to **touch**, or be **tangent** to, the circle at that point. The point is called the **point of contact**, or **point of tangency**, and the line is called a **tangent**.

A *tangent from a point to a circle* is to be understood as the segment of the tangent between the point and the circle.

If the two points in which a secant cuts a circumference continually approach, the secant approaches the condition of tangency. Hence the tangent is sometimes spoken of as *a secant in its limiting position.*

Corollaries. 1. *One, and only one, tangent can be drawn to a circle at a given point on the circumference.*

For the tangent is perpendicular to the radius at that point, and there is only one such perpendicular. (Has this been proved ?)

2. *Any tangent is perpendicular to the radius drawn to the point of contact.* (Why ?)

3. *A line perpendicular to a radius at its extremity on the circumference is tangent to the circle.* (Why ?)

4. *The center of a circle lies on the perpendicular to any tangent at the point of contact.*

For the radius to that point is perpendicular to the tangent, and as there is only one such perpendicular at that point (prel. prop. II), that perpendicular must be the radius.

5. *The perpendicular from the center to a tangent meets it at the point of contact.*

For the radius to that point is perpendicular to the tangent, and there is only one perpendicular from the center to the tangent.

Exercises. 272. Show that of these three properties of a line, (1) the passing through the center of a circle, (2) the being perpendicular to a given chord, (3) the bisecting of that chord, any two in general necessitate the third. In what special case is there an exception ?

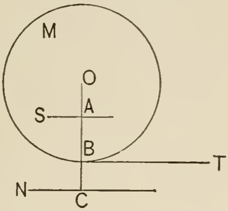
273. If a chord is bisected by a second chord, and the second by a third, and the third by a fourth, and so on, the points of bisection approach nearer and nearer the center.

274. Tangents drawn to a circle from the extremities of a diameter are parallel.

275. The diameter of a circle bisects all chords which are parallel to the tangent at either extremity.

PROPOSITION X.

190. Theorem. *An unlimited straight line cuts a circumference, touches the circle, or does not meet the circle, according as its distance from the center of the circle is less than, equal to, or greater than, the radius.*



Given OA, OB, OC, the perpendiculars from center O of $\odot M$, to lines S, T, N, and respectively less than, equal to, greater than, the radius.

To prove that S is a secant, T a tangent, N a line not meeting M.

Proof. 1. A, B, C are respectively within the $\odot$, on the circumference, or without the $\odot$. § 108, def. $\odot$, cor. 3

2. $\therefore S$ is a secant. § 109, 2

3. And T is a tangent. Prop. IX, cor. 3

4. $N \parallel T$. I, prop. XVI, cor. 3

5. And $\therefore N$ cannot meet $\odot M$ because it cannot cross T.

COROLLARY. *The converses are true.*

Let the student state this corollary in full, and show that the Law of Converse (§ 73) applies.

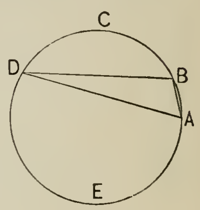
Exercises. 276. What is the locus of the extremities of equal tangents drawn from points on a circumference?

277. Two tangents meet at a point the length of a diameter distant from the center of the circle. How many degrees in their included angle?

3. ANGLES FORMED BY CHORDS, SECANTS, AND TANGENTS.

191. Definitions. A **segment of a circle** is either of the two portions into which the circle is cut by a chord.

If a segment is not a semicircle, it is called a **major** or a **minor segment** according as its arc is a major or minor arc.



E.g. DBC is a minor segment, and BDE is a major segment.

The fact that the word *segment* is used to mean a part of a line, and also a part of a circle, will not present any difficulty, since the latter use is rare, and the sense in which the word is used is always evident. It means "a part cut off," and is therefore applicable to both cases.

192. The angle, not reflex, formed by two chords which meet on the circumference is called an **inscribed angle,** and is said to *stand upon*, or *be subtended by, the arc* which lies within the angle and is cut off by the arms.

It is also called an *angle inscribed in*, or simply an *angle in, the segment* whose arc is the conjugate of the arc on which it stands.

$\angle ADB$ is an inscribed angle, standing on $\widehat{AB}$; it is also an angle in the segment $BCDEA$. Similarly, $\angle DBA$ is in the segment DAC and stands on $\widehat{DA}$.

193. Points lying on the same circumference are called **concyclic.**

Exercises. 278. If from the extremities of any chord perpendiculars to that chord are drawn, they will cut off equal segments measured from the extremities of any diameter. (Draw a perpendicular from the center to the chord.)

279. If a tangent from a point B on a circumference meets two tangents from A, C, on the circumference, in points X, Y; and if the lines joining the center to A, X, Y, C, are a, x, y, c, respectively, then $\angle xy = \angle ax + \angle yc$, and $XY = AX + YC$.

PROPOSITION XI.

194. Theorem. *An inscribed angle equals half the central angle standing on the same arc.*

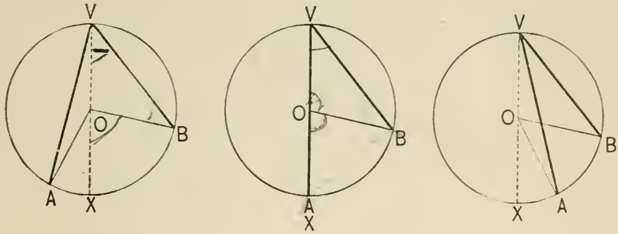


FIG. 1. FIG. 2. FIG. 3.

Given AVB an inscribed angle, and AOB the central angle on the same arc AB.

To prove that $\angle AVB = \frac{1}{2} \angle AOB$.

Proof. 1. Suppose VO drawn through center O, and produced to meet the circumference at X.

Then $\angle XVB = \angle VBO$. I, prop. III

2. And $\angle XOB = \angle XVB + \angle VBO$, Why?

$= 2 \angle XVB$. Step 1

3. $\therefore \angle XVB = \frac{1}{2} \angle XOB$. Ax. 7

4. Similarly $\angle AVX = \frac{1}{2} \angle AOX$ (each = zero in Fig. 2),

and $\therefore \angle AVB = \frac{1}{2} \angle AOB$. Ax. 2

The proof holds for all three figures, point A having moved to X (Fig. 2), and then through X (Fig. 3).

195. The theorem is often stated thus: *An inscribed angle is measured by half its intercepted arc.*

This expression, like that mentioned in § 180 is not strictly correct. The angle and the arc simply have the same *numerical* measure as proved later in § 254.

Corollaries. 1. *Angles in the same segment, or in equal segments, of a circle are equal.* (Why?)

2. *If from a point on the same side of a chord as a given segment, lines are drawn to the ends of that chord, the angle included by those lines is greater than, equal to, or less than, an angle in that segment, according as the point is within, on the arc of, or without, the segment.*

This follows from cor. 1 and from I, prop. IX. Draw the figure and prove.

3. *The converse of cor. 2 is true by the Law of Converse. Hence the locus of the vertex of a constant angle whose arms pass through two fixed points is an arc.*

Let the student state the converse in full, and give the proof.

Exercises. 280. In the figures on p. 129, prove that if P is taken anywhere on $\widehat{BV}$, then $\angle PBV + \angle BVP$ is constant.

281. In Fig. 3, p. 129, if BO is produced to meet the circumference at W, and the point of intersection of BW and AV is called Y, prove that $\triangle s$ YVB and WAY are mutually equiangular.

282. What is the locus of the vertex of a triangle on a given base and with a given vertical angle? Prove it.

283. In Fig. 1, p. 129, suppose A to move freely on the arc $VAXB$, and suppose $\angle s$ AVB, VBA bisected by lines meeting at P. Show that the locus of P is a constant arc.

284. If the vertices of a hexagon are concyclic, the sum of any three alternate interior angles is a perigon. (That is, the sum of three angles, taking every other one.)

285. Two equal chords with a common extremity are symmetric with respect to the diameter through that extremity, as an axis; so also are their corresponding arcs.

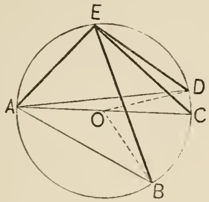
286. If from any point P, on the diameter AB, PX and PY are drawn to the circumference on the same side of AB and making $\angle XPA = \angle BPY$, then $\triangle s$ APX and YPB are mutually equiangular.

287. If any number of triangles on the same base and on the same side of it have equal vertical angles, the bisectors of the angles are concurrent.

288. Prove that two chords perpendicular to a third chord at its extremities are equal.

PROPOSITION XII.

196. Theorem. *An angle in a segment is greater than, equal to, or less than, a right angle, according as the segment is less than, equal to, or greater than, a semicircle.*



Given the segments ADE, ACE, ABE of a circle with center O, respectively less than, equal to, greater than, a semicircle.

To prove that $\angle s$ AED, AEC, AEB are respectively greater than, equal to, less than, a right angle.

Proof. 1. Draw OB, OD.

Then $\because \angle AED = \frac{1}{2}$ reflex $\angle AOD$, Prop. XI

$\therefore \angle AED >$ rt. $\angle$.

2. And $\because \angle AEC = \frac{1}{2}$ st. $\angle AOC$, Why?

$\therefore \angle AEC =$ rt. $\angle$.

3. And $\because \angle AEB = \frac{1}{2}$ oblique $\angle AOB$, Why?

$\therefore \angle AEB <$ rt. $\angle$.

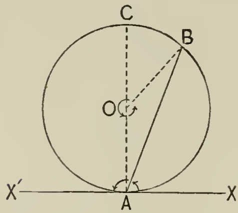
COROLLARY. *A segment is less than, equal to, or greater than, a semicircle, according as the angle in it is greater than, equal to, or less than, a right angle.*

From prop. XII, by the Law of Converse, § 73. Let the student write out the proof.

NOTE. The discovery that an angle in a semicircle is a right angle is attributed to Thales, who, tradition asserts, sacrificed an ox to the gods in honor of the event.

Proposition XIII.

197. Theorem. *An angle formed by a tangent and a chord of a circle equals half of the central angle standing on the intercepted arc.*



Given AB a chord, XX' a tangent through A, and O the center of the circle.

To prove that $\angle XAB = \frac{1}{2} \angle AOB$,

and $\angle BAX' = \frac{1}{2} \angle BOA$.

Proof. 1. Produce AO to meet the circumference at C.

Then $\angle XAC = \frac{1}{2} \angle AOC$,

$= \frac{1}{2}$ st. $\angle$. Why?

2. And $\because \angle BAC = \frac{1}{2} \angle BOC$, Why?

$\therefore \angle XAB = \frac{1}{2} \angle AOB$. Ax. 3

3. Also, $\because \angle CAX' = \frac{1}{2} \angle COA$, Why?

$\therefore \angle BAX' = \frac{1}{2} \angle BOA$. Ax. 2

Corollary. *Tangents to a circle from the same external point are equal.*

For, connect the points of tangency, and two angles of the triangle are equal by this theorem.

198. The theorem is often stated thus: *An angle formed by a tangent and a chord of a circle is measured by half its intercepted arc.*

See § 195.

Dec. 8

PROPOSITION XIV.

199. Theorem. *An angle formed by two unlimited intersecting lines which meet the circumference equals either the sum or the difference of half the central angles on the intercepted arcs, according as the point of intersection is within or without the circle.*

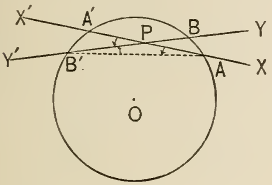


FIG. 1.

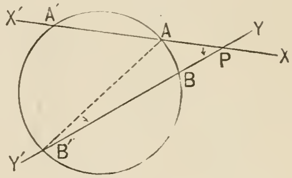


FIG. 2.

Given two lines XX', YY' meeting a circumference at A, A' and B, B', respectively, and intersecting at P.

To prove that $\angle A'PB'$ equals half the central angle on $A'B'$ plus or minus half that on AB, according as P is within or without the circle.

Proof. Suppose AB' drawn.

Then $\angle A'PB' = \angle A'AB' \pm \angle AB'B$. § 88

$= \frac{1}{2}$ cent. $\angle$ on $\widehat{A'B'} \pm \frac{1}{2}$ cent. $\angle$ on $\widehat{AB}$.

Prop. XI

The theorem is thus re-stated for two of the special cases:

COROLLARIES. 1. *An angle formed by two chords equals the sum of half the central angles on the intercepted arcs.*

See Fig. 1. (State this as suggested in § 195.)

2. *An angle formed by two secants intersecting without the circle equals the difference of half the central angles on the intercepted arcs.*

See Fig. 2. (State this as suggested in § 195.)

Prop. XIV is, of course, true for tangents as well as chords and secants. The following figures represent special cases.

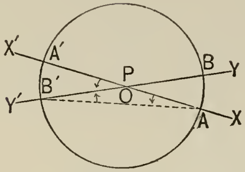


Fig. 3.

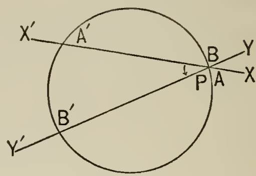


Fig. 4.

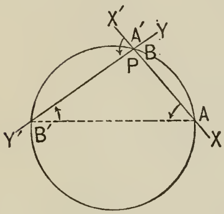


Fig. 5.

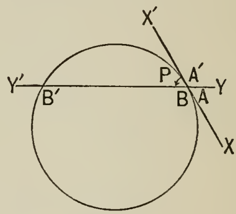


Fig. 6.

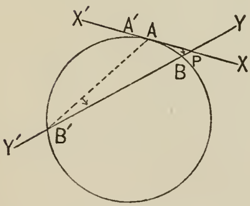


Fig. 7.

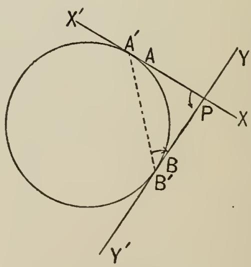


Fig. 8.

Fig. 3 is a special case where P is at O, and merely affirms that a central angle equals itself. Fig. 4 shows that prop. XI is a special case of prop. XIV. Fig. 6 shows the same for prop. XIII.

Corollary. 3. *An angle formed by a secant and a tangent, or by two tangents, equals the difference of half the central angles on the intercepted arcs.*

See Figs. 7 and 8.

PROPOSITION XV.

200. Theorem. *If two parallel lines intercept arcs on a circumference, those arcs are equal.*

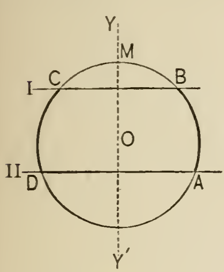


FIG. 1.

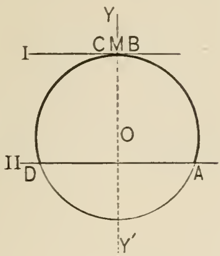


FIG. 2.

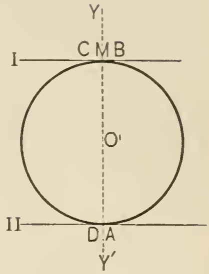


FIG. 3.

Given two parallel lines, I and II, intercepting arcs AB, CD, on the circumference of a circle with center O.

To prove that $\widehat{AB} = \widehat{CD}$.

Proof. 1. Suppose $YOY' \perp I$, and to cut $\widehat{BC}$ at M, Fig. 1.

Then $YY' \perp II$. I, prop. XVII, cor. 1

2. And $\widehat{BM} = \widehat{MC}$, and $\widehat{AM} = \widehat{MD}$. Prop. V

3. $\therefore \widehat{AB} = \widehat{CD}$. Ax. 3

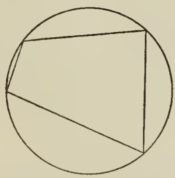
NOTE. The proof is the same for Figs. 2, 3; in Fig. 2, $\widehat{BC}$ equals zero; and in Fig. 3, $\widehat{DA}$ also equals zero. It should be noticed that Figs. 1, 2, 3, respectively, may be considered as special, or at least as limiting cases of Figs. 2, 7, and 8 of prop. XIV. In prop. XIV as P moves farther and farther to the right the lines come nearer and nearer to being parallel, the angle APB approaches nearer and nearer zero, and hence the central angles on arcs BA, $A'B'$ approach nearer and nearer equality. It might therefore be inferred that when the lines become parallel, the arcs become equal, as proved in prop. XV.

Exercise. 289. The chords which join the extremities of two equal arcs are either parallel, or else they intersect and are equal and cut off equal segments from each other.

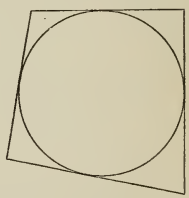
4. INSCRIBED AND CIRCUMSCRIBED TRIANGLES AND QUADRILATERALS.

201. Definitions. If the vertices of the angles of a polygon lie on a circumference, the polygon is said to be **inscribed in the circle,** and the circle is called a **circumscribed circle.**

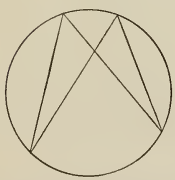
If the lines of the sides of a polygon are tangent to a circle, the polygon is said to be **circumscribed about the circle,** and the circle is called an **inscribed** or **escribed circle,** according as it lies within or without the polygon.



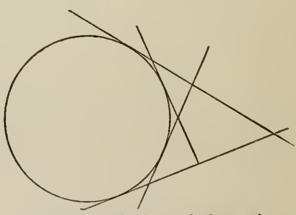
Inscribed quadrilateral.
Circumscribed circle.



Circumscribed quadrilateral.
Inscribed circle.



Inscribed cross quadrilateral.
Circumscribed circle.



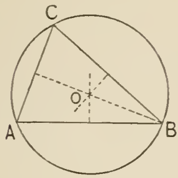
Circumscribed quadrilateral.
Escribed circle.

The words *inscriptible*, *circumscriptible*, *escriptible* mean capable of being inscribed in, circumscribed about, escribed to, a circle.

Exercise. 290. If any two chords cut within the circle, at right angles, the sum of the squares on their segments equals the square on the diameter.

PROPOSITION XVI.

202. Theorem. *A circumference can be described to pass through the three vertices of any triangle.* (*Circumscribed circle.*)



Given the points A, B, C, the vertices of $\triangle ABC$.

To prove that a circumference can be described to pass through A, B, C.

Proof. 1. There is such a circumference. § 131, cor. 2

2. And the center of the ⊙ can be found. § 131, cor. 1

NOTE. The relation between prop. XVI and prop. XVII should be noticed. Similarly for props. XVIII and XIX, and for XX and XXI.

Exercises. 291. Prove from prop. XVI and prop. XI that the sum of the interior angles of any triangle equals a straight angle.

292. If the hypotenuse of a right-angled triangle is the diameter of a circle, the circumference passes through the vertex of the right angle. (COROLLARY. *The median from the vertex of the right angle of a right-angled triangle equals half of the hypotenuse.*)

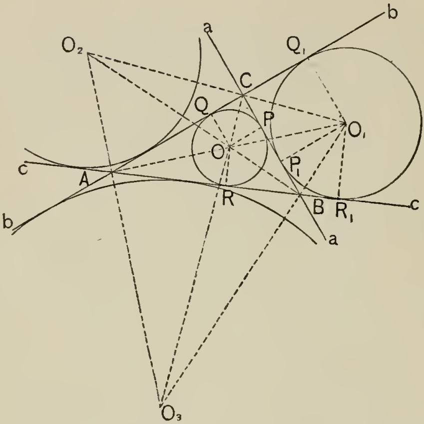
293. A line-segment of constant length slides so as to have its extremities constantly resting on two lines perpendicular to each other. Find the locus of its mid-point.

294. If a circle is described on the line joining the orthocenter to any vertex, as a diameter, prove that the circumference passes through the feet of the perpendiculars from the other vertices to the opposite sides.

295. Prove that the perpendiculars from the vertices of a triangle to the opposite sides bisect the angles of the triangle formed by joining their feet; the so-called *Pedal Triangle.*

Proposition XVII.

203. Theorem. *A circle can be described tangent to the three lines of any triangle. (Inscribed and escribed circles.)*



Given the lines a, b, c, forming a $\triangle ABC$.

To prove that a circle can be described tangent to a, b, c.

Proof. 1. Let O be the in-center, O_1, O_2, O_3 the ex-centers.

Let OP, OQ, $OR \perp a$, b, c.

Then $\triangle ARO \cong \triangle AQO$,

and $\triangle BRO \cong \triangle BPO$. I, prop. XIX, cor. 7

2. $\therefore OQ = OR = OP$. Why?

3. $\therefore P$, Q, R are concyclic. § 108, def. ⊙, cor. 3

4. And $\because AB \perp OR$, AB is a tangent. Prop. IX, cor. 3

Similarly, a, b, c, are tangent to the other three Ⓢ.

Corollary. *A circle can be described tangent to three lines not all parallel nor concurrent.*

PROPOSITION XVIII.

204. Theorem. *In an inscribed quadrilateral the sum or difference of two opposite angles equals the sum or difference of the other two opposite angles, according as the quadrilateral is convex or cross.*

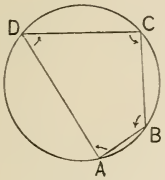


FIG. 1.

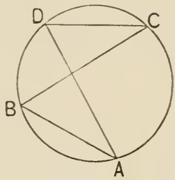


FIG. 2.

Given the inscribed convex quadrilateral $ABCD$.

To prove that in Fig. 1, $\angle A + \angle C = \angle B + \angle D$.

Proof for Fig. 1. 1. $\angle A + \angle C = \frac{1}{2}$ central $\angle$ on $\widehat{BD} + \widehat{DB}$
$= \text{st.} \angle$. Prop. XI

2. Similarly, $\angle B + \angle D = \text{st.} \angle$.

3. $\therefore \angle A + \angle C = \angle B + \angle D$. § 30

Proof for Fig. 2. If the quadrilateral is cross, $\angle C - \angle A = \angle D - \angle B$, since each equals zero. Why?

COROLLARIES. 1. *A parallelogram inscribed in a circle has all of its angles equal, and is therefore a rectangle.* (Why?)

2. *The opposite angles of an inscribed convex quadrilateral are supplemental.*

Exercises. 296. In the figure of prop. XIII, if P is the mid-point of arc AB, prove that P is equidistant from AX and AB. Suppose the arc BCA is taken, instead of AB.

297. If a circle is described on one side of a triangle as a diameter, prove that the circumference passes through the feet of the perpendiculars drawn to the other two sides from the opposite vertices.

Proposition XIX.

205. Theorem. *In a circumscribed quadrilateral the sum or difference of two opposite sides equals the sum or difference of the other two opposite sides, according as the quadrilateral is convex or cross.*

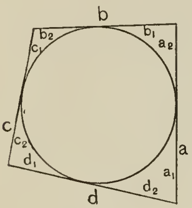


Fig. 1.

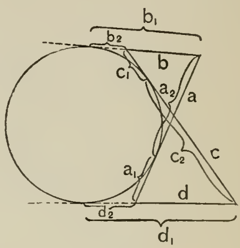


Fig. 2.

Given the circumscribed convex quadrilateral *abcd*.

To prove that in Fig. 1, $a + c = b + d$.

Proof for Fig. 1, as lettered.

1. $a_1 = d_2$, $a_2 = b_1$, $c_1 = b_2$, $c_2 = d_1$. Prop. XIII, cor.
2. $\therefore a_1 + a_2 + c_1 + c_2 = b_1 + b_2 + d_1 + d_2$. Ax. 2
3. Or, $a + c = b + d$. Ax. 8

Proof for Fig. 2. If the quadrilateral is cross, $c - a = d - b$.

1. $\because c_1 = b_2$, and $c_2 = d_1$, $\therefore c = b_2 + d_1$.
2. $\because a_1 = d_2$, and $a_2 = b_1$, $\therefore a = b_1 + d_2$.
3. $\therefore c - a = d - b$. Ax. 3

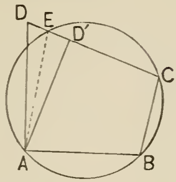
Corollary. *A parallelogram circumscribed about a circle has all of its sides equal, and is therefore a rhombus.* (Why?)

Exercises. 298. The bisector of an angle formed by a tangent and chord bisects the intercepted arc.

299. Given two pairs of parallel chords, $AB \parallel A'B'$, and $BC \parallel B'C'$; prove that $AC' \parallel A'C$.

PROPOSITION XX.

206. Theorem. *If the sum of two opposite angles of a quadrilateral equals the sum of the other two opposite angles, the quadrilateral is inscriptible.*



Given the quadrilateral $ABCD$ such that

$$\angle A + \angle C = \angle B + \angle D.$$

To prove that $ABCD$ is inscriptible.

Proof. 1. Suppose the circumference determined by A, B, C not to pass through D, but to cut CD at E. Prop. XVI
Draw AE. Then $\angle B + \angle AEC = \angle C + \angle BAE$.
Prop. XVIII

2. But $\angle B + \angle D = \angle C + \angle A$.
and $\therefore \angle AEC - \angle D = \angle BAE - \angle A$,
or, $\angle EAD = - \angle EAD$. I, prop. XIX
But this is absurd; hence step 1 is absurd.
The proof is the same for D'.

COROLLARY. *If two opposite angles of a quadrilateral are supplemental, the quadrilateral is inscriptible.*

Exercises. **300.** A square is inscriptible.

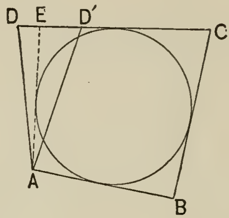
301. Every equiangular quadrilateral is inscriptible.

302. The intersection of the diagonals of an equiangular quadrilateral is the center of the circumscribed circle.

303. A circle is described on one of the equal sides of an isosceles triangle as a diameter. Prove that the circumference bisects the base.

Proposition XXI.

207. Theorem. *If the sum of two opposite sides of a quadrilateral equals the sum of the other two opposite sides, the quadrilateral is circumscriptible.*



Given the quadrilateral $ABCD$ such that

$$AB + CD = BC + DA.$$

To prove that $ABCD$ is circumscriptible.

Proof. 1. Suppose the ⊙ tangent to AB, BC, CD not to be tangent to DA, but to be tangent to EA. Prop. XVII

Then $AB + CE = BC + EA$. Prop. XIX

2. But $AB + CD = BC + DA$, Given

and $\therefore$ $CD - CE$, or ED, $= DA - EA$. Ax. 3

But this is absurd; hence step 1 is absurd.

I, prop. VIII, cor.

The proof is the same for D'.

Exercises. **304.** A square is circumscriptible. (Notice the relation between exs. 300–302 and exs. 304–306.)

305. Every equilateral quadrilateral is circumscriptible.

306. The intersection of the diagonals of an equilateral quadrilateral is the center of the inscribed circle.

307. A', B' are the feet of perpendiculars from A, B on a, b in $\triangle ABC$; M is the mid-point of AB. Prove that $\angle B'A'M = \angle MB'A' = \angle C$.

5. TWO CIRCLES.

208. Definitions. Two circles are said to **touch** or to be **tangent** when their circumferences have one, and only one, point in common.

They are said to be *internally* or *externally tangent* according as one circle lies within or without the other. The more accurate expression, *a tangent circumference*, is often used instead of *a tangent circle.*

The line determined by the centers of two circles is called their **center-line**; the segment of the center-line, between the centers, is called their **center-segment.**

If two circles have a common center, they are said to be **concentric.**

The expression *concentric circumferences* is also used.

Exercises. 308. A triangle is inscribed in a circle. Prove that the sum of three angles, one in each segment of the circle, exterior to the triangle, equals a perigon.

309. Prove that a perpendicular from the orthocenter of a triangle to a side, produced to the circumference of the circumscribed circle, is bisected by that side.

310. Prove that the bisectors of any angle of an inscribed quadrilateral and the opposite exterior angle meet on the circumference.

311. If the diagonals of an inscribed quadrilateral bisect each other, what kind of a quadrilateral is it?

312. Prove that if two consecutive sides of a convex hexagon inscribed in a circle are respectively parallel to their opposite sides, the remaining sides are parallel to each other.

313. Prove that the bisectors of the angles formed by producing the opposite sides of an inscribed quadrilateral to meet, are perpendicular to each other. (A proof may be based on cors. 1 and 2 of prop. XIV.)

314. Prove that if the diagonals of an inscribed quadrilateral are perpendicular to each other, the line through their intersection perpendicular to any side bisects the opposite side. (Brahmagupta's theorem.)

PROPOSITION XXII.

209. Theorem. *If two circumferences meet in a point which is not on their center-line, then (1) they meet in one other point, (2) their center-line is the perpendicular bisector of their common chord, (3) their center-segment is greater than the difference and less than the sum of the radii.*

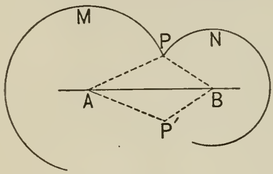


FIG. 1.

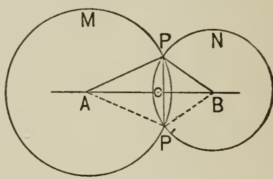


FIG. 2.

Given M and N, two circumferences with centers A, B, meeting at P not on AB.

To prove that (1) they meet again, as at P';

(2) $AB \perp PP'$ and bisects it, as at C;

(3) $AB >$ the difference between AP and BP and $< AP + BP$.

Proof. 1. In Fig. 1, suppose $\triangle ABP$ revolved about AB as an axis of symmetry, thus determining $\triangle AP'B$.

Then $\because AP' = AP$, and $BP' = BP$,

$\therefore P'$ is on both M and N, which proves (1).

§ 108, def. $\odot$, cor. 3

2. In Fig. 2, $\because AP = AP'$, and $BP = BP'$, § 109, 1

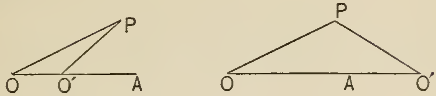
3. $\therefore A$ and B lie on the $\perp$ bisector of PP', which proves (2). I, prop. XLI

4. $AB >$ the difference between AP and BP and $< AP + BP$, which proves (3). § 75 and cor.

COROLLARY. *If two circumferences meet at one point only, that point is on their center-line.* (Why?)

PROPOSITION XXIII.

210. Theorem. *If two circles meet on their center-line, they are tangent.*



Given O and O', the centers of two circles with radii OA, $O'A$, which meet on their center-line at A.

To prove that the circles are tangent.

Proof. 1. Let P be any point, other than A, on circumference with center O, and draw OP, $O'P$.

Then $OO' + O'P > OP$ or its equal OA. I, prop. VIII

2. And $\because OO' = OA - O'A$,

$\therefore OA - O'A + O'P > OA$,

or $O'P > O'A$,

by adding $O'A$ and subtracting OA. Axs. 4, 5

3. $\therefore P$ is without the circle with center O'. § 108, def. ⊙, cor. 3

4. And $\because$ the Ⓢ have only one point in common, $\therefore$ they are tangent. § 208

COROLLARIES. 1. *If two circumferences intersect, neither point of intersection is on the center-line.* (Why?)

2. *If two circles touch, they have a common tangent-line at the point of contact.*

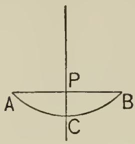
For a perpendicular to their center-line at that point is tangent to both. (Why?)

Exercise. 315. Find the locus of the centers of all circles tangent to a given circle at a given point.

6. PROBLEMS.

Proposition XXIV.

211. Problem. *To bisect a given arc.*



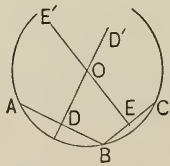
Solution. 1. Draw its chord AB. § 28

2. Draw $PC \perp AB$ at its mid-point. §§ 114, 116

Then PC bisects the arc. Prop. V, cor. 2

Proposition XXV.

212. Problem. *To find the center of a circle, given its circumference or any arc.*



Given a circumference, or an arc ABC.

Required to find the center of the circle.

Solution. 1. Draw two chords from B, as BA, BC. § 28

2. Draw their $\perp$ bisectors DD', EE', §§ 114, 116

intersecting at the center O. § 131, cors. 1, 4

Note. Hereafter it will be assumed that the center is known if an arc is known, for it may always be found by this problem.

PROPOSITION XXVI.

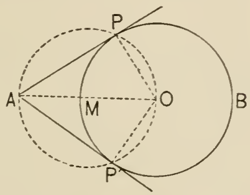
213. Problem. *To draw a tangent to a given circle from a given point.*

1. *If the point is on the circumference.*

Solution. 1. At the given point erect a perpendicular to the radius drawn to the point. I, prop. XXIX

2. This is the required tangent, and the solution is unique. Prop. IX, cors. 3, 1

2. *If the point is without the circle.*



Given a circle $PP'B$, with center O; also an external point A.

Required from A to draw a tangent to $\odot\ PP'B$.

Construction. 1. Draw AO. § 28

2. Bisect AO at M. I, prop. XXXI

3. Describe a $\odot$ with center M, radius MO. § 109

4. Join A to intersections of circumferences. § 28

Then these lines, AP, AP', are the required tangents.

Proof. 1. The circumferences will have two points in common, and only two. Prop. XXII; I, prop. XLIII, cor. 3

2. And $\because \measuredangle s\ APO,\ OP'A$ are rt. $\measuredangle s$, Why?

$\therefore AP,\ AP'$ are tangents. Why?

(Would this solution hold for case 1?)

Proposition XXVII.

214. Problem. *To draw a common tangent to two given circles.*

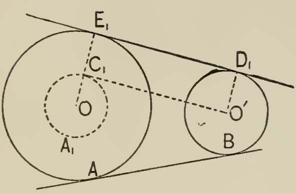


Fig. 1.

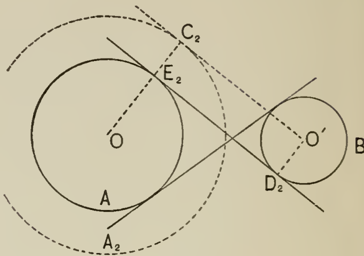


Fig. 2.

Given two circles A, B, with radii r, r' $(r > r')$, and centers O, O', respectively.

Required to draw a common tangent to them.

Construction. 1. Describe $\circledS\ A_1, A_2$ (Figs. 1 and 2), with centers O, and radii $r - r'$ and $r + r'$, respectively. § 109

2. From O' draw tangents $O'C_1$, $O'C_2$, to $\circledS\ A_1, A_2$. Prop. XXVI

3. Draw OC_1, OC_2, cutting circumferences A at E_1, E_2. § 28

4. Draw $O'D_1 \parallel OE_1$, and $O'D_2 \parallel E_2O$. § 118

5. Draw E_1D_1, E_2D_2; they are the tangents.

Proof. 1. $\angle s\ C_1, C_2$ are rt. $\angle s$. Why?

2. In Fig. 1,

$\because C_1E_1 = OE_1 - OC_1 = r - (r - r') = r'$,

$\therefore C_1E_1 =$ and $\parallel O'D_1$. Const. 1, 4

3. $\therefore C_1O'D_1E_1$ is a $\square$, and $\angle s\ E_1, D_1$ are rt. $\angle s$, I, props. XXV, XXIII, cor.

and $\therefore D_1E_1$ is tangent to $\circledS\ A, B$. Prop. IX, cor. 3

Similarly, in Fig. 2, $E_2C_2 =$ and $\parallel D_2O'$, and $E_2D_2O'C_2$ is a $\square$, and D_2E_2 is a tangent. In both figures a second tangent can evidently be drawn, the solution being analogous to that above given. Hence there are four tangents in general.

NOTE. The following special cases are of interest.

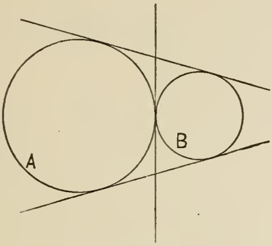


FIG. 3.

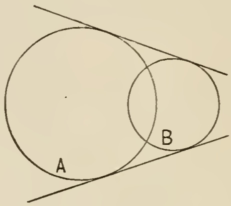


FIG. 4.

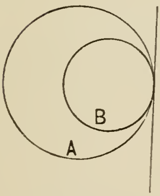


FIG. 5.

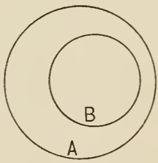


FIG. 6.

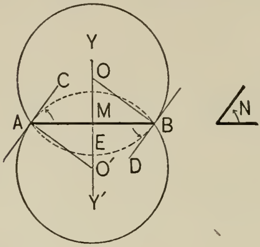
In Fig. 3 the two circles have moved to external tangency, and the two interior tangents have closed up into one. In Fig. 4 the circumferences intersect and the interior tangents have vanished. In Fig. 5 the circles have become internally tangent and the two exterior tangents have closed up into one. In Fig. 6 the circle B lies wholly within the circle A, and the tangents have all vanished. In all cases the center-line is evidently an axis of symmetry.

Exercises. **316.** All tangents drawn from points on the outer of two concentric circumferences to the inner are equal.

317. Find the locus of the centers of all circles touching two intersecting lines. (Show that it is a pair of perpendiculars.) Suppose the two lines were parallel instead of intersecting.

Proposition XXVIII.

215. Problem. *On a given line-segment as a chord to construct a segment of a circle containing a given angle.*



Given the line-segment AB and the $\angle N$.

Required on AB to construct a segment of a circle, containing $\angle N$.

Construction. 1. Draw BD and AC, making ∠s ABD, BAC equal to $\angle N$. I, prop. XXXII

2. Draw YY', the $\perp$ bisector of AB. I, prop. XXXI

3. Draw $\perp$'s to AC, BD, from A, B. I, prop. XXIX

These $\perp$'s will intersect YY' at the centers of the ⓢ whose segments on AB are required.

Proof. 1. The two $\perp$'s from A, B, meet YY', as at O', O. I, prop. XVII, cor. 4

2. O is the center of ⊙ with chord AB and tangent BD. Prop. V, cor. 2; prop. IX, cor. 4

3. $\therefore \angle ABD$, or $\angle N$, $= \frac{1}{2}$ central $\angle$ on $\widehat{AEB}$, Prop. XIII

$= \angle$ in segment ABY. Prop. XI

Similarly for segment $Y'BA$, where $\angle BAC = \frac{1}{2}$ central $\angle$ on the intercepted arc.

216. Definitions. Two intersecting arcs are said to form an angle, meaning thereby the angle included by their respective tangents at the point of intersection.

An arc and a secant are said to form an angle, meaning thereby the angle included by the secant and the tangent to the arc at the point of meeting.

E.g. in the figure of prop. XXVIII, OB is said to make a right angle with the circumference EBA, because it is perpendicular to the tangent at B.

Exercises. 318. The bisectors of the interior and the exterior vertical angles of a triangle meet the circumscribed circumference in the mid-points of the arcs into which the base divides that circumference, and the line joining those points is the diameter which bisects the base.

319. A triangle whose angles are, respectively, 30°, 50°, 100° is inscribed in a circle; the bisectors of the angles meet the circumference in A, B, C. Find the number of degrees in the angles of $\triangle ABC$.

320. The three sides of $\triangle ABC$ are, respectively, 412 in., 506 in., 514 in.; required the lengths of the six segments formed by the three points of tangency of the inscribed circle.

321. The radii of two concentric circles are 29 in. and 36 in., respectively. In the larger circle a chord is drawn tangent to the smaller; required its length.

322. Two circumferences of circles of radii 0.5 ft. and 1.2 ft. intersect so that the tangents drawn at their point of intersection are perpendicular to each other. Required the distance between the centers.

323. The distance between the centers of two circles of radii 7 in. and 4 in., respectively, is 8 in. Required the length of their common tangent, between the points of tangency. Is there more than one answer?

324. The distance between the centers of two circles of radii 327 in. and 115 in., respectively, is 729 in. Required the length of their common exterior tangent, between the points of tangency.

325. The distance between the centers of two circles is 165 in.; the radii are 62 in. and 48 in., respectively. Calculate, correct to 0.001, the length of the longest line parallel to the center-line and 30 in. from it, limited by the circumferences.

326. Through the point A, 6 in. from the center of a circle of radius 4.5 in., two tangents, AT, AT', are drawn. Calculate the length of the chord TT' and its distance from the center.

APPENDIX TO BOOK III. — METHODS.

217. The student has already been informed of three important methods of attacking a proposition:

(1) By Analysis (§ 113).
(2) By Intersection of Loci (I, props. XLIII, XLIV).
(3) By *Reductio ad Absurdum* (§ 74).

He is now prepared to discuss these somewhat more fully.

218. I. Method of Analysis. This method, first found in Euclid's Geometry, though attributed to Plato, may be thus described: Analysis is a kind of inverted solution; *it assumes the proposition proved, considers what results follow, and continues to trace these results until a known proposition is reached.* It then seeks to *reverse the process* and to give the usual, or *Synthetic*, proof.

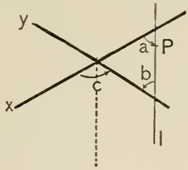
A more modern form of analysis is sometimes known as the Method of Successive Substitutions. In this the student substitutes in place of the given proposition another upon which the given one depends, and so on until a familiar one is reached. The student reasons somewhat as follows:

1. I can solve A if I can solve B.
2. And I can solve B if I can solve C.
3. But I *can* solve C.

Or he reasons thus:

1. A is true if B is true.
2. And B is true if C is true.
3. But C *is* true.
4. Hence A and B are true.

ILLUSTRATIVE EXERCISES. 1. *Through a given point to draw a line to make equal angles with two intersecting lines.*



Analysis. Suppose x, y the lines, P the point, and l the required line; then, in the figure, $\angle c = \angle a + \angle b$; but $\because \angle a$ is to equal $\angle b$, $\therefore \angle c = 2 \angle a$; $\therefore$ if $\angle c$ is bisected, and a line is drawn through P parallel to this bisector, the construction is effected. Now that the method is discovered, *give the solution in the ordinary way.*

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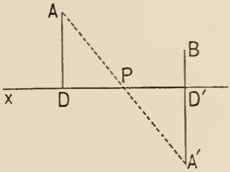
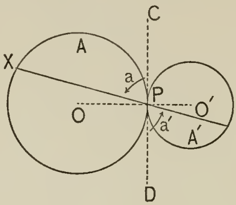
2. *Through a given point to draw a line such that the segments intercepted by the perpendiculars let fall upon it from two given points shall be equal.*



Analysis. Suppose P the given point through which the line x is to be drawn, and A and B the other given points; then, in the figure, AD and $BD' \perp x$, and DP is to equal PD'. Further, if AP is produced to meet BD' produced at A', then $\triangle DPA \cong \triangle D'PA'$, and $\therefore AP = PA'$. But $\because A$ and P are given, AP can be drawn, and PA' found; $\therefore A'$ can be found, and $\therefore A'B$; then from P a $\perp$ can be drawn to $A'B$, and the problem is solved. Always give the solution in the ordinary way.

3. *If two circles are tangent, any secant drawn through their point of contact cuts off segments from one that contain angles equal to the angles in the segments of the other.*



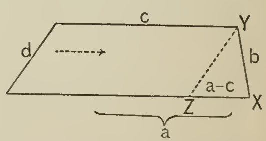
Analysis. 1. Let CD be the common tangent to Ⓢ O, O' at their point of contact P. III, prop. XXIII, cor. 2

2. Then an $\angle$ in segment A = an $\angle$ in segment A', *if*

$\angle a = \angle a'$. III, prop. XIII

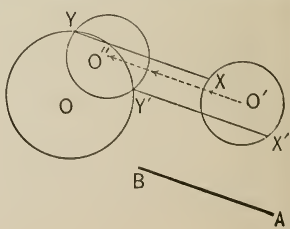
3. But $\angle a = \angle a'$. Prel. prop. VI

Exercises. **327.** To construct a trapezoid, given the four sides.



Analysis. Assume the figure drawn. Then if d is moved parallel to itself and between c and a, to the position YZ, the $\triangle XYZ$ can easily be constructed (I, prop. XXXIV). The process may now be reversed and the trapezoid constructed.

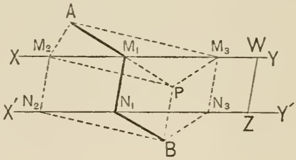
328. To place a line so that its extremities shall rest upon two given circumferences, the line being equal and parallel to another line.



Analysis. If O and O' are the given circles, and AB the given line, and if ⊙ O' is moved along a line parallel and equal to AB, then either XY or $X'Y'$ answers the conditions. Hence the process may be reversed; first describe ⊙ O'', and then from Y, Y' draw YX and $Y'X'$ = and $\parallel$ BA.

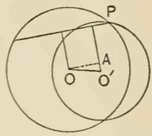
EXERCISES.

329. Given two parallels, XY, $X'Y'$, with a transversal WZ limited by XY and $X'Y'$; also two points A, B, not between the parallels, and on opposite sides of them. Required to join A and B by the shortest broken line which shall have MN, the intercept between XY and $X'Y'$, parallel to WZ.



Analysis. If any MN in the figure is moved along NB parallel to its original position, until N coincides with B and M is at P, then $AM_1P <$ AM_2P or AM_3P (I, prop. VIII); hence AM_1N_1B is the shortest broken line. Hence the process may be reversed; first draw $BP \parallel$ and $= ZW$; then join A and P, thus fixing M_1; and then draw $M_1N_1 \parallel WZ$.

330. Through one of the two points of intersection of two circumferences to draw a line from which the two circumferences cut off chords having a given difference. (The projection of the center-segment on the required line equals half the given difference; hence move this projection to the position OA; the right-angled $\triangle\ OO'A$ can now be constructed, and the required line will be parallel to OA.)



331. In ex. 330, show that if the two chords lie on opposite sides of P, the *sum* replaces the *difference*.

332. In a given circle to draw a chord equal and parallel to a given line.

333. From a ship two known points are seen under a given angle; the ship sails a given distance in a given direction, and now the same two points are seen under another known angle. Find the positions of the ship. (On the line joining the known points, construct segments to contain the given angles; the problem then reduces to ex. 328.)

334. Construct a trapezoid, given the diagonals, their included angle, and the sum of two adjacent sides.

335. To construct a triangle given a and the orthocenter.

336. Also, given a and the centroid.

337. To draw a tangent to a given circle, perpendicular to a given line.

338. To construct a triangle, ABC, having given c, $\angle C$, and the foot of the perpendicular from C to c.

339. Find the locus of the points of contact of tangents drawn from a fixed point to a system of concentric circles.

219. II. Method of Intersection of Loci. This method, adapted chiefly to the solution of problems, has already been used in Book I (props. XLIII, XLIV). So long as it is known merely that a point is on *one* line, its position is not definitely known; but if it is known that the point is also on another line, its position *may* be uniquely determined. For example, if it is known that a point is on each of two intersecting lines, the point is uniquely determined as their point of intersection; but if the point is on a straight line and a circumference which the line intersects, it may be either of the two points of intersection.

For convenience of reference the following theorems are stated, and will be referred to by the letters prefixed:

a. The locus of points at a given distance from a given point is the circumference described about that point as a center, with a radius equal to the given distance. (§ 127.)

b. The locus of points at a given distance from a given line consists of a pair of parallels at that distance, one on each side of the fixed line. (§ 129, cor. 2.)

c. The locus of points equidistant from two given points is the perpendicular bisector of the line joining them. (§ 128.)

d. The locus of points equidistant from two given lines consists of the bisectors of their included angles; if the lines are parallel, it is a parallel midway between them. (§ 129.)

e. The locus of points from which a given line subtends a given angle is an arc subtended by that chord. (§ 195, cor. 3.)

Abbreviations. The following abbreviations will be used: In the triangle ABC the altitudes on the sides a, b, c will be designated by h_a, h_b, h_c, respectively; the corresponding medians by m_a, m_b, m_c; the corresponding angle-bisectors terminated by a, b, c, by v_a, v_b, v_c; the radii of the inscribed and circumscribed circles by r, R, respectively; the radius of the escribed circle touching a, and touching b and c produced, by r_a, and similarly for r_b, r_c.

220. DEFINITION. A *triangle is said to be inscribed in another* when its vertices lie respectively on the sides of the other.

Exercises. 340. To describe a circumference with a given radius, and

(1) Passing through two given points. (Combine a and c.)

(2) Passing through one given point and touching a given line. (a, b.)

(3) Passing through one given point and touching a given circle. (a.)

(4) Touching a given line and a given circle. (a, b.)

(5) Touching two given circles. (a.)

341. In a given triangle to inscribe a triangle with two of its sides given, and the vertex of their included angle given. (a.)

342. To describe a circumference passing through a given point and touching a given line, or a given circle, in a given point. (c.)

343. On a given circumference to find a point having a given distance from a given line. (b.)

344. On a given line, not necessarily straight, to find a point equidistant from two given points. (c.)

345. Describe a circumference touching two parallel lines and passing through a given point. (d, a.)

346. Find a point from which two given line-segments are seen under (or subtend) given angles. (e.) (Pothenot's problem.)

347. Construct the triangle ABC, given a, h_a, m_a.

348. Also, given $\angle A$, a, h_a.

349. Also, given $\angle A$, a, m_a.

350. Also, given a, h_b, h_c.

351. Also, given $\angle A$, h_a, v_a. (First construct the right-angled triangle with side h_a and hypotenuse v_a.)

352. Also, given h_a, m_a, R. (First construct the right-angled triangle with side h_a and hypotenuse m_a; then find the circumcenter by a, c.)

353. Also, given a, R, h_b. (First construct the right-angled triangle with side h_b and hypotenuse a; then find the circumcenter by a.)

354. Also, given c, r, $\angle A = 90°$; c, r_c, $\angle A = 90°$; b, r, $\angle A = 90°$; or b, r_c, $\angle A = 90°$.

355. Describe two circles of given radii r_1, r_2, to touch one another, and to touch a given line on the same side of it.

MISCELLANEOUS EXERCISES.

356. If two circumferences intersect, any two parallel lines drawn through the points of intersection and terminated by the respective circumferences are equal.

357. If the center-segment of two circles is (1) greater than, (2) equal to, the sum of the two radii, the circumferences (1) do not meet, (2) are tangent.

358. The greatest of all lines joining two points, one on each of two given circumferences, is greater than the center-segment by the sum of the radii.

359. If two circles, whose centers are O, O', are tangent at P, and a line through P cuts the circumferences at A, A', prove that $OA \parallel O'A'$. Two cases; external and internal tangency. Show that the proposition is true for any number of circles.

360. Through a vertex of a triangle to draw a straight line equally distant from the other vertices.

361. Describe a circle of given radius to touch two given lines. Show that a solution is, in general, impossible if the lines are parallel, but that otherwise there are four solutions.

362. From what two points in the plane are two circles seen under equal angles?

363. Given an equilateral triangle, ABC, find a point P such that the circles circumscribing $\triangle$ PBC, PCA, PAB are all equal.

364. To divide a circle into two segments so that the angle contained in one shall be double that contained in the other.

365. From two given points to draw lines meeting a given line in a point and making equal angles with that line, the points being on (1) the same side of the given line, (2) opposite sides of the given line.

366. To draw, through a given point, a secant from which two equal circumferences shall cut off equal chords. Discuss the number of solutions for various positions of the given point.

367. Through one of the points of intersection of two circumferences to draw a chord of one circle which shall be bisected by the circumference of the other.

368. Two opposite vertices of a given square move on two lines at right angles to each other. Find the locus of the intersection of the diagonals.

369. Find the locus of the intersection of two lines passing through two fixed points on a circumference and intercepting an arc of constant length.

BOOK IV. — RATIO AND PROPORTION.

1. FUNDAMENTAL PROPERTIES.

221. Introductory Note. The inference was drawn in Book II (§ 155) that a relation exists between algebra and geometry with the following correspondence:

Geometry.	*Algebra.*
A line-segment.	A number.
The rectangle of two line-segments.	The product of two numbers.

And as it was assumed that a straight line may be represented by a number, so it may be assumed that any other geometric magnitude, such as an arc, an angle, a surface, etc., may be represented by a number. With these assumptions, the fundamental properties of Ratio and Proportion may be proved either by algebra or by geometry, as may be most convenient, the proof being valid for both of these subjects. The purely geometric treatment is too difficult for the beginner.

222. Definitions. To measure a magnitude is to find how many times it contains another magnitude of the same kind, called the **unit of measure.**

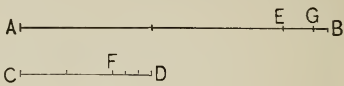
223. A **ratio** is the quotient of the numerical measure of one magnitude divided by the numerical measure of another magnitude of the same kind.

For example, the ratio of a line 8 ft. long to one 16 ft. long is $\frac{8}{16}$, or $\frac{1}{2}$; that of one 16 ft. long to one 8 ft. long is 2.

The ratio of a to b is expressed by the symbols $\frac{a}{b}$, $a : b$, a/b, or $a \div b$.

If the ratio $\frac{a}{b} = r$, then $a = r \cdot b$.

224. *The practical method of finding the ratio* of two magnitudes is to measure them, and to divide the numerical result of one measurement by that of the other. But if two line-segments have a common measure, their ratio and their common measure may be found by the following process:



Let AB and CD be the two lines.

Apply CD as often as possible to AB, and suppose that

$$AB = 2\ CD + EB,\ \ EB < CD.$$

Similarly, apply EB to CD, and suppose that

$$CD = 2\ EB + FD,\ \ FD < EB.$$

Similarly, apply FD to EB, and suppose that

$$EB = FD + GB,\ \ GB < FD.$$

Similarly, apply GB to FD, and suppose that

$$FD = 3\ GB \text{ with no remainder.}$$

Then $\quad FD = 3\ GB.$

$$EB = FD + GB = 4\ GB.$$

$$CD = 2\ EB + FD = 8\ GB + 3\ GB = 11\ GB.$$

$$AB = 2\ CD + EB = 22\ GB + 4\ GB = 26\ GB.$$

$\therefore$ GB is a common measure, and the ratio of CD to AB is $\frac{11}{26}$ by definition of ratio.

225. Definitions. Two magnitudes that have a common measure are said to be **commensurable**; if they have no common measure they are said to be **incommensurable.**

For example, two surfaces having areas 10 sq. in. and 15 sq. in. are said to be commensurable, there being the common measures 5 sq. in., 1 sq. in., 2.5 sq. in., etc. But if the length of one line is represented by $\sqrt{2}$, and the length of another by 1, then there is no common measure, and the lines are said to be incommensurable.

A ratio may therefore be an integer, or a fraction, or an irrational number such as $\sqrt{2}$.

For practical purposes all magnitudes may be looked upon as commensurable, since a unit of measure can be so taken that the remainder may be made as small as we wish.

226. In the ratio $a : b$, a and b are called the **terms** of the ratio, — the former, a, being called the **antecedent**, and the latter, b, the **consequent**.

227. If the ratio $a : b$ equals the ratio $c : d$, the four terms are said to form a **proportion**.

The four terms are also said to be *in proportion*. The terms a and b are also said to be *proportional* to c and d.

This equality of ratios is indicated by the symbol $=$, *e.g.*, $\frac{a}{b} = \frac{c}{d}$, $a : b = c : d$, or $a/b = c/d$, read "a is to b as c is to d." Instead of the parallel bars ($=$), the double colon ($::$) is also used in this connection as a sign of equality, the proportion being written $a : b :: c : d$. The double colon is not, however, as extensively used as formerly.

228. The first and last terms of a proportion are called the **extremes**, and the other terms the **means**.

Thus in the proportion $2 : 3 = 6 : 9$, 3 and 6 are the means and 2 and 9 are the extremes.

PROPOSITION I.

229. Theorem. *If* $a : b = c : d$, *then* $ad = bc$.

Proof. From $\frac{a}{b} = \frac{c}{d}$, it follows, by multiplying equals by bd,

that $ad = bc$. Ax. 6

Hence

If four *numbers* are in proportion, the *product* of the means equals the *product* of the extremes.

If four *lines* are in proportion, the *rectangle* of the means equals the *rectangle* of the extremes.

Proposition II.

230. Theorem. *If* ad = bc, *then* a : b = c : d.

Proof. Divide the given equals by *bd*. Ax. 7

Hence

If the product of two *numbers* equals the *product* of two other *numbers*, either two may be made the means and the other two the extremes of a proportion.

If the *rectangle* of two *lines* equals the *rectangle* of two other *lines*, either two may be made the means and the other two the extremes of a proportion.

Proposition III.

231. Theorem. *If* a : b = c : d, *then* (1) a : c = b : d, (2) d : b = c : a, *and* (3) b : a = d : c.

Proof. 1. $ad = bc$, Prop. I

and $\therefore \frac{a}{c} = \frac{b}{d}$, which proves (1), Prop. II

and $\frac{d}{b} = \frac{c}{a}$, which proves (2). Prop. II

2. And $\because bc = ad$, Step 1

3. $\therefore \frac{b}{a} = \frac{d}{c}$, which proves (3). Prop. II

Hence

If four *numbers* are in proportion, the following interchanges may be made: (1) the means, (2) the extremes, (3) each antecedent and its corresponding consequent.

If four *magnitudes* are in proportion, the following interchanges may be made: (1) the means, (2) the extremes, (3) each antecedent and its corresponding consequent.

Definitions. The proportion $a : c = b : d$ is often spoken of as the proportion $a : b = c : d$ *taken by* **alternation.**

The proportion $b : a = d : c$ is also spoken of as the proportion $a : b = c : d$ *taken by* **inversion.**

Hence prop. III may be stated: *If four magnitudes are in proportion they are in proportion by alternation and also by inversion.*

But to take a proportion by alternation, the magnitudes must be similar. Thus \$2 : \$4 = \$8 : \$16, therefore, by alternation, \$2 : \$8 = \$4 : \$16. But the proportion \$2 : \$4 = 8 days : 16 days cannot be taken by alternation, for \$2 : 8 days = \$4 : 16 days means nothing, \$2 : 8 days not being a ratio (§ 223).

PROPOSITION IV.

232. Theorem. *If* $\mathrm{a : b = c : d}$,

then (1) $\mathrm{ka : b = kc : d}$,

and (2) $\mathrm{a : kb = c : kd}$.

Proof. From $\frac{a}{b} = \frac{c}{d}$, it follows, by multiplying by k,

that $\frac{ka}{b} = \frac{kc}{d}$, which proves (1). Ax. 6

And also, $\frac{a}{kb} = \frac{c}{kd}$, by dividing by k, which proves (2). Ax. 7

Hence if four magnitudes are in proportion, and both antecedents or both consequents are multiplied by the same number, the magnitudes are still in proportion.

COROLLARY. *If four magnitudes are in proportion, and all are multiplied by the same number, the results are in proportion.*

NOTE. The number k may be integral, fractional, or irrational.

Proposition V.

233. Theorem. *If* a : b = c : d,

then (1) $a \pm b : b = c \pm d : d$,

(2) $a \pm b : a = c \pm d : c$,

and (3) $a \pm b : a \mp b = c \pm d : c \mp d$.

Proof. 1. From $\frac{a}{b} = \frac{c}{d}$, it follows

that $\frac{a}{b} \pm 1 = \frac{c}{d} \pm 1$, Axs. 2, 3

or $\frac{a \pm b}{b} = \frac{c \pm d}{d}$, which proves (1).

2. It is also true

that $\frac{b}{a} = \frac{d}{c}$, Prop. III

and $\therefore 1 \pm \frac{b}{a} = 1 \pm \frac{d}{c}$, Axs. 2, 3

or $\frac{a \pm b}{a} = \frac{c \pm d}{c}$, which proves (2).

3. Or, by subtracting first,

$\frac{a \mp b}{a} = \frac{c \mp d}{c}$, Axs. 3, 2

and $\therefore \frac{a \pm b}{a \mp b} = \frac{c \pm d}{c \mp d}$ by dividing in the last two equations. Ax. 7

The proportion $a + b : b = c + d : d$ is often spoken of as the proportion $a : b = c : d$ *taken by* **composition,** and $a - b : b = c - d : d$ as the same proportion *taken by* **division,** and $a \pm b : a \mp b = c \pm d : c \mp d$ as the same proportion *taken by* **composition and division.**

PROPOSITION VI.

234. Theorem. *If* $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \cdots\cdots$, *the terms all being magnitudes of the same kind, then*

$$\frac{a_1 + a_2 + \cdots\cdots}{b_1 + b_2 + \cdots\cdots} = \frac{a_1}{b_1} \text{ or } \frac{a_2}{b_2} \text{ or } \cdots\cdots$$

Proof. 1. $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, Prop. III

and $\therefore \frac{a_1 + a_2}{a_2} = \frac{b_1 + b_2}{b_2}$. Prop. V

2. $\therefore \frac{a_1 + a_2}{b_1 + b_2} = \frac{a_2}{b_2}$ Prop. III

$= \frac{a_3}{b_3} = \cdots\cdots$ Given

3. Then, as in steps 1, 2.

$$\frac{a_1 + a_2 + a_3}{b_1 + b_2 + b_3} = \frac{a_3}{b_3} = \cdots\cdots$$

and so on, however many ratios there may be.

PROPOSITION VII.

235. Theorem. *If* $a : b = c : d$, *then* $a^2 : b^2 = c^2 : d^2$.

Proof. $\because \frac{a}{b} = \frac{c}{d}$, $\therefore \frac{a^2}{b^2} = \frac{c^2}{d^2}$. Ax. 6

Hence

If four *numbers* are in proportion, *their squares* are also in proportion.

If four *lines* are in proportion, the *squares on those lines* are also in proportion.

COROLLARY. *If* $a : b = c : d$, *and* $m : n = x : y$, *then* $am : bn = cx : dy$.

Proposition VIII.

236. Theorem. $\mathrm{a : b = ka : kb}$.

Proof. $kab \equiv kab$, or $a \cdot kb = b \cdot ka$.

$\therefore a : b = ka : kb$. Prop. II

Note. The number k may be integral, fractional, or irrational.

237. Definitions. If $a : b = c : x$, x is called the **fourth proportional** to a, b, c.

Corollaries. 1. *By three of the four terms of a proportion the other is determined.*

For if $a : b = c : x$, or $x : b = c : a$, or $c : x = a : b$, etc., it follows that $ax = bc$, whence $x = bc/a$, a fixed number.

2. *If* $\mathrm{a : b = a : x}$, *then* $\mathrm{b = x}$.

For if, in the proof of cor. 1, $c = a$, then $b = x$.

238. If, in a proportion, the two means are equal, as in $a : x = x : b$, this common mean is called the **mean proportional,** or **geometric mean,** between the two extremes.

Corollaries.

The mean proportional *between two* numbers *equals the* square root *of their* product.

The geometric mean *between two* lines *equals the* side of that square *which equals their* rectangle.

Because the number representing the square units of area of a rectangle is the product of the two numbers representing the linear units in two adjacent sides, the expression *product of two lines* is often used for *rectangle of two lines.*

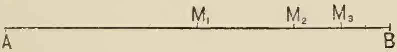
Exercises. 370. Find a mean proportional between 2 and 32.

371. Find a fourth proportional to 3, 7, 15.

372. What number must be added to each of the numbers 2, 1, 5, 3, to have the results in proportion ?

2. THE THEORY OF LIMITS.

239. Definitions. A quantity is called a **variable** if, in the course of the same investigation, it may take indefinitely many values; on the other hand, a quantity is called a **constant** if, in the course of the same investigation, it keeps the same value.



E.g. if a line AB is bisected at M_1, and M_1B at M_2, and M_2B at M_3, and so on, and if x represents the line from A to any of the points M_1, M_2. ....., then x is a variable, but AB is a constant.

It is customary, as in algebra, to represent variables by the last letters of the alphabet, and constants by the first letters.

240. If a variable x approaches nearer and nearer a constant a, so that the difference between x and a can become and remain smaller than any quantity that may be assigned, then a is called the **limit** of x.

E.g. in the above figure, AB is the limit of x.

But if the point M simply slides along the line, passing through B, then, although the difference between AM and AB, or x and a, can *become* smaller than any quantity which may be assigned, it does not *remain* smaller, for when M passes through B this difference increases. Hence AB is not then the limit of x.

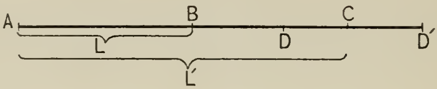
That "x approaches as its limit a" is indicated by the symbol $x \doteq a$.

COROLLARY. *If* $x \doteq a$, *then* $a - x$ *is a variable whose limit is zero; that is,* $a - x \doteq 0$.

Although the variable has been taken, in this discussion, as *increasing* towards its limit, it may also be taken as *decreasing*. Thus if we bisect a line, bisect its half, and continue to bisect indefinitely, the variable segment is evidently approaching a limit zero.

Proposition IX.

241. Theorem. *If, while approaching their respective limits, two variables have a constant ratio, their limits have that same ratio.*



Given X and X', two variables, such that as they *increase* they approach their respective limits AB, or L, and AC, or L', and have a constant ratio r.

To prove that $L : L' = r$, or that $X : X' = L : L'$.

Proof. If the ratio $X : X'$ is not equal to the ratio $L : L'$, then (1) it must equal the ratio of L to something less than L', or (2) it must equal the ratio of L to something greater than L'.

It will be shown that both of these suppositions are absurd.

I. *To show that* (1) *is absurd.*

1. Suppose $X : X' = L : L' - DC$.

   Then $\because X : X' = r$,

   $\therefore X = rX'$,

   and $L = r(L' - DC)$. § 223, def. ratio

2. Then $L - X = r(L' - DC - X')$. Ax. 3

3. But $\because X' \doteq L'$,

   $\therefore L' - X'$ can become as small as we please.

   $\therefore$ " " " less than DC,

   and $\therefore r(L' - X' - DC)$ can become negative.

4. But $\because X \not> L$, $\therefore L - X$ cannot become negative.

5. $\therefore$ step 2 is absurd, for a negative quantity cannot equal one not negative.

II. *To show that* (2) *is absurd.*

1. Suppose $X : X' = L : L' + CD'$.
   Then $L - X = r(L' + CD' - X')$,
   as in step 2, p. 168.

2. But $r(L' + CD' - X')$ cannot become less than $r \cdot CD'$.

3. And $L - X \doteq 0$, because L is the limit of X.

4. $\therefore$ if step 1 were true, a quantity, $L - X$, which can become as small as we please, would equal a quantity not less than $r \cdot CD'$, which is absurd.

The proof would be substantially the same if the two variables were supposed to *decrease* toward a limit.

COROLLARIES. 1. *If, while approaching their respective limits, two variables are always equal, their limits are equal.*

For their ratio is always 1.

2. *If, while approaching their respective limits, two variables have a constant ratio, and one of them is always greater than the other, the limit of the first is greater than the limit of the second.*

For the limits have the same ratio as the variables.

Exercises. 373. If $a : b = c : d$,

prove that $a^2bd + b^2c + bc = ab^2c + abd + ad.$

1. Since $bc = ad,$ Prop. I

the equation is true if $a^2bd + b^2c = ab^2c + abd.$

2. Now if in place of each ad we put bc, we see that the equation is true if $ab^2c + b^2c = ab^2c + b^2c.$

But this is an identity. Hence the proof is complete.

374. If $a : b = c : d$,

prove that $a - c : b - d = \sqrt{a^2 + c^2} : \sqrt{b^2 + d^2}.$

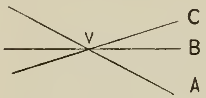
Also that $\sqrt{a^2 + c^2} : \sqrt{b^2 + d^2} = \sqrt{ac + \frac{c^3}{a}} : \sqrt{bd + \frac{d^3}{b}}.$

Also that $a + mb \cdot a - nb = c + md : c - nd.$

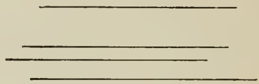
3. A PENCIL OF LINES CUT BY PARALLELS.

242. Definitions. Through a point any number of lines can be passed. Such lines are said to form a **pencil of lines.**

The point through which a pencil of lines passes is called the **vertex** of the pencil.



A pencil of three lines.



A pencil of four parallels.

The annexed pencil of three lines is named "$V - ABC$."

To conform to the idea of a general figure, set forth in §§ 94, 95, the word *pencil* is also applied to parallel lines, the vertex being spoken of as "at infinity."

243. Definition. Two lines are said to be **divided proportionally** when the segments of the one have the same ratio as the corresponding segments of the other.

Exercises. 375. If $a : b = c : d$, prove that

(1) $a + b + c + d : b + d = c + d : d.$

(2) $m(a + mb) : n(a - nb) = m(c + md) : n(c - nd).$

(3) $a(a + b + c + d) = (a + b)(a + c).$

(4) $a^2c + ac^2 : b^2d + bd^2 = (a + c)^3 : (b + d)^3.$

376. If b is a mean proportional between a and c, prove that

$$\frac{a^2 - b^2 + c^2}{\frac{1}{a^2} - \frac{1}{b^2} + \frac{1}{c^2}} = b^4.$$

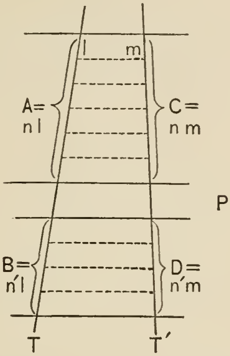
377. Show that there is no finite number which, when added to each of four unequal numbers in proportion, will make the resulting sums in proportion.

378. If $a : b = c : d$, and $u : v = x : y$,

prove that $au + bv : au - bv = cx + dy : cx - dy.$

PROPOSITION X.

244. Theorem. *The segments of a transversal of a pencil of parallels are proportional to the corresponding segments of any other transversal of the same pencil.*



Given the pencil of parallels P, cutting from two transversals T and T' the segments A, B and C, D, respectively.

To prove that $A : B = C : D$.

Proof. 1. Suppose A and B divided into equal segments l, and that $A = nl$, while $B = n'l$.

(In the figure, $n = 6$, $n' = 4$.)

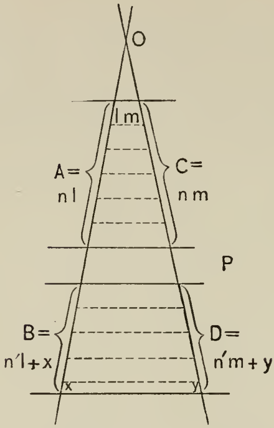
Then if ∥'s to P are drawn from the points of division, C is the sum of n equal segments m, and D is the sum of n' equal segments m.

I, prop. XXVII, cor. 1

2. $$\therefore \frac{A}{B} \equiv \frac{nl}{n'l} = \frac{n}{n'} = \frac{nm}{n'm} \equiv \frac{C}{D}.$$ Why?

NOTE. The preceding proof assumes that A and B are commensurable. The following proof is valid if A and B are incommensurable.

245. **Proof for incommensurable case.**



1. Suppose A divided into equal segments l, and that

$$A = nl,$$

while $B = n'l +$ some remainder, x, such that $x < l$. Then if ‖'s to P are drawn from the points of division, C is the sum of n equal segments m, and D is the sum of n' equal segments m, + a remainder y such that $y < m$.

2. Then B lies between $n'l$ and $(n'+1)\,l$. Step 1

3. $\therefore \dfrac{B}{A}$ lies between $\dfrac{n'l}{nl}$ and $\dfrac{(n'+1)\,l}{nl}$

$\left(\text{in the figure, between } \dfrac{4\,l}{6\,l} \text{ and } \dfrac{5\,l}{6\,l}\right)$,

while $\dfrac{D}{C}$ lies between $\dfrac{n'm}{nm}$ and $\dfrac{(n'+1)\,m}{nm}$.

4. $\therefore \dfrac{B}{A}$ and $\dfrac{D}{C}$ both lie between $\dfrac{n'}{n}$ and $\dfrac{n'+1}{n}$,

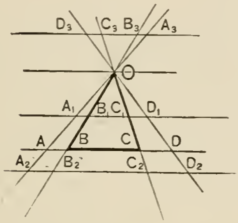
and $\therefore$ they differ by less than $\dfrac{1}{n}$.

(In the figure, by less than $\frac{1}{6}$.)

5. And $\because \frac{1}{n}$ can be made smaller than any assumed difference, by increasing n, $\therefore$ to assume any difference leads to an absurdity.

6. $$\therefore \frac{B}{A} = \frac{D}{C}, \text{ or } \frac{A}{B} = \frac{C}{D}.$$ Prop. III

COROLLARIES. 1. *A line parallel to one side of a triangle divides the other two sides proportionally.*



For in the figure, if BCO is the triangle, the lines OB, OC are cut by parallels. Hence $BB_1 : B_1O = CC_1 : C_1O$.

2. *The corresponding segments of the lines of a pencil cut off (from the vertex) by parallel transversals are proportional.*

In the above figure, $OA : OA_1 = OB : OB_1 = OC : OC_1 = \ldots\ldots$, by prop. X.

3. *The segments of the lines of a pencil cut off (from the vertex) by parallel transversals are proportional to the corresponding segments of the transversals.*

To prove that, in the above figure, $AB : A_1B_1 = OA : OA_1 = OB : OB_1$.

Draw through A_1 a line ∥ to OB cutting AB at X.

Then $\quad AB : XB = OA : OA_1 = OB : OB_1$. Prop. X

But $\quad XB = A_1B_1$. I, prop. XXIV

4. *Parallel transversals are divided proportionally by the lines of a pencil.*

To prove that, in the above figure, $AB : BC = A_1B_1 : B_1C_1$.

By cor. 3, $AB : A_1B_1 = BO : B_1O = BC : B_1C_1$. Hence, etc.

Proposition XI.

246. Theorem. *A line can be divided, internally or externally, into segments having a given ratio, except that if it is divided externally the ratio cannot be unity.*

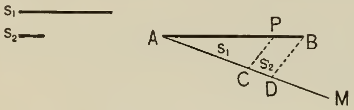


Fig. 1.

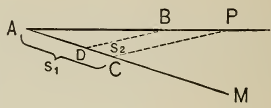


Fig. 2.

Given the line AB, and two lines s_1, s_2 having a given ratio.

To prove that AB can be divided in the ratio $s_1 : s_2$, except that in the case of external division s_1 cannot equal s_2.

Proof. 1. Suppose AM drawn making, with AB, an angle $< 180°$; that AC be taken $= s_1$, and $CD = s_2$; that DB be drawn, and $CP \parallel DB$.

2. Then $AP : PB = s_1 : s_2$, as required. Prop. X, cor. 1

3. In Fig. 2, if $s_1 = s_2$, where does D fall? What is then the relation of CP to AB? Hence show that the division is impossible in this case.

Corollary. *The point of internal division is unique; likewise the point of external division.*

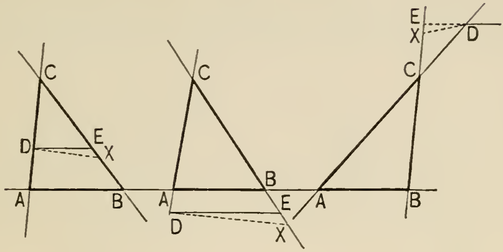
From step 2, $AB : PB = s_1 + s_2 : s_2$, AB, $s_1 + s_2$, and s_2, all being constants; but by three terms of a proportion the fourth is determined. (§ 237, def. of 4th proportional, cor. 1.)

247. Note. Instead of saying that the external division, if the ratio is unity, is impossible, it is often said that the point of division, P, is *at infinity*.

In the case of internal division, the ratios $AP : PB$ and $AC : CD$ are evidently positive; but in the case of external division each ratio is evidently negative because PB and CD are negative. In both cases step 2 is evidently true.

Proposition XII.

248. Theorem. *A line which divides two sides of a triangle proportionally is parallel to the third.*



Given the triangle ABC, and DE so drawn that $AD : DC = BE : EC$.

To prove that $DE \parallel AB$.

Proof. 1. Suppose DE not $\parallel AB$, but that $DX \parallel AB$.

Then $BX : XC = AD : DC$. Prop. X, cor. 1

2. But this is impossible, for the division of BC in the ratio $AD : DC$ is unique. Prop. XI, cor.

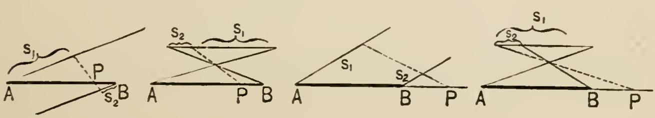
3. $\therefore DX$ must be identical with DE, and $DE \parallel AB$.

The proof is the same for all of the figures.

Exercises. 379. In the above figures, if $AD : DC = BE : EC = m : n$, and if the line through A and E cuts the line through B and D at P, then prove that $AP : PE = BP : PD = m + n : n$.

380. If ex. 379 has been proved, show from it that the centroid of a triangle divides the medians in the ratio of $2 : 1$.

381. Prove prop. XI on the following figures:



Proposition XIII.

249. Theorem. *If any angle of a triangle is bisected, internally or externally, by a line cutting the opposite side, then the opposite side is divided, internally or externally, respectively, in the ratio of the other sides of the triangle.*

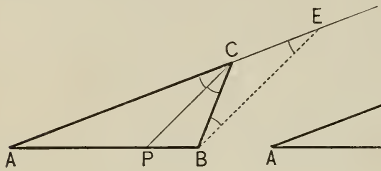


Fig. 1.

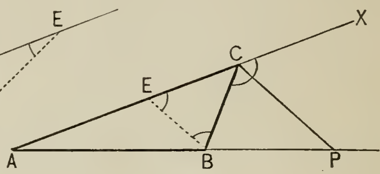


Fig. 2.

Given $\triangle ABC$, the bisector of $\angle C$ meeting AB at P.

To prove that $AP : PB = AC : BC$.

Proof. 1. Let $BE \parallel PC$, meeting AC produced at E, in Fig. 1.

Then $\angle EBC = \angle PCB = \angle ACP = \angle CEB$.

Given; I, prop. XVII, cor. 2

2. $\therefore BC = CE$. Why?

3. But in $\triangle ABE$, $AP : PB = AC : CE$, Prop. X, cor. 1

and $\therefore AP : PB = AC : BC$. Why?

The proof for Fig. 2 is the same if step 1 is changed to $\angle CBE = \angle BCP = \angle PCX = \angle BEC$.

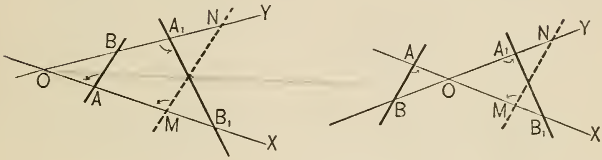
250. Definition. When a line is divided internally and externally into segments having the same ratio, it is said to be divided **harmonically.**

If the internal and external points of division of AB, in prop. XIII, are P and P', then AB is divided harmonically by P and P'.

Exercise. 382. The hypotenuse of a right-angled triangle is divided harmonically by any pair of lines through the vertex of the right angle, making equal angles with one of its arms.

4. A PENCIL CUT BY ANTIPARALLELS OR BY A CIRCUMFERENCE.

251. Definitions. If a pencil of two lines $O - XY$ is cut by two parallel lines AB, MN, and if MN revolves, through a



straight angle, about the bisector of $\angle XOY$ as an axis, falling in the position A_1B_1, then AB and A_1B_1 are said to be **antiparallel** to each other.

OA and OA_1 are called *corresponding segments* of the pencil, as are also OB and OB_1. A and A_1 are called *corresponding points*, as are also B and B_1.

COROLLARY. *If $\angle A = \angle A_1$, in the above figure, then AB and A_1B_1 are antiparallel to each other.*

Exercises. 383. From P, a given point in the side AB of $\triangle ABC$, draw a line to AC produced so that it will be bisected by BC.

384. Investigate ex. 383 when P is on AB produced.

385. If the vertices of $\triangle XYZ$ lie on the sides of $\triangle abc$ so that $x \parallel a$, $y \parallel b$, $z \parallel c$, then X, Y, Z bisect a, b, c.

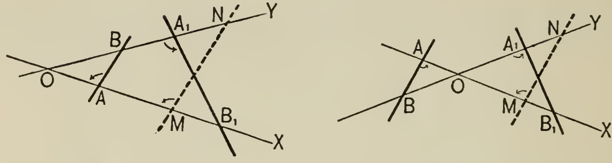
386. In prop. XIII, suppose $\angle B = \angle A$; also, suppose $\angle B < \angle A$.

387. In any triangle the line joining the feet of the perpendiculars from any two vertices to the opposite sides is antiparallel to the third side.

388. In $\triangle ABC$, suppose that $a \perp c$, and the bisectors of the interior and exterior angles at C meet AB at P_1, P_2. Prove that if a circumference passes through P_1, P_2, and C, (1) P_1P_2 is the diameter, (2) AC is a tangent.

Proposition XIV.

252. Theorem. *If a pencil of two lines is cut by two antiparallel lines, the corresponding segments form a proportion.*



Given the pencil $O - XY$, cut by the antiparallels AB, A_1B_1, A and A_1 being corresponding points.

To prove that $OA : OA_1 = OB : OB_1$.

Proof. 1. Suppose MN the parallel to AB which, revolving, fixed A_1B_1.

Then $OA_1 = OM$, and $OB_1 = ON$. Def. antipar. § 251

2. But $OA : OM = OB : ON$. Prop. X, cor. 2

and $\therefore OA : OA_1 = OB : OB_1$. Substitution

Corollary. *If two antiparallels cut a pencil of two lines, the product of the segments of one line equals the product of the segments of the other.*

Why? What is meant by "product of two segments"?

Exercises. 389. In the above figures, $AB : A_1B_1 = OA : OA_1 = OB : OB_1$. (Prop. X, cor. 3, etc.)

390. In the above figures, if A_1 coincides with B, and if $OB = b$, $OA = a$, $OB_1 = b_1$, then $b^2 = ab_1$.

391. If from the vertex of a right-angled triangle a perpendicular p is drawn cutting the hypotenuse c into two segments x, y, adjacent to sides a, b, respectively, then (1) a and p are antiparallels of the pencil b, c; (2) a is a mean proportional between c and x; (3) p is a mean proportional between x and y; (4) $b^2 = cy$, $a^2 = cx$, and $\therefore a^2 + b^2 = c(x + y) = c^2$. (Thus a new proof is found for the Pythagorean proposition.)

PROPOSITION XV.

253. Theorem. *If a pencil of lines cuts a circumference, the product of the two segments from the vertex is constant, whichever line is taken.*

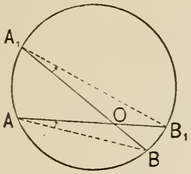


FIG. 1.—The point O on the chord.

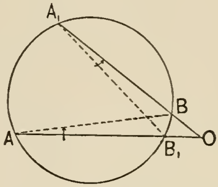


FIG. 2.—The point O on the chord produced.

Given AB_1 and A_1B, two chords, each divided at O into two segments.

To prove that $AO \cdot OB_1 = A_1O \cdot OB.$

Proof. 1. Suppose AB, A_1B_1 drawn.

Then $\angle A = \angle A_1$. Why?

2. $\therefore AB$ and A_1B_1 are antiparallel, § 251, cor.

and $\therefore AO \cdot OB_1 = A_1O \cdot OB.$ Prop. XIV, cor.

The proposition is entirely general and should be proved for the following cases.

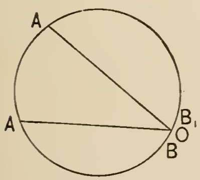


FIG. 3.—The point O at the end of the chord.

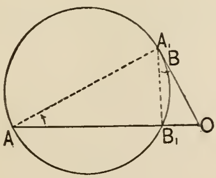


FIG. 4.—Chord A_1B becomes zero.

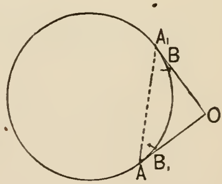


FIG. 5.—Chord AB_1 also becomes zero.

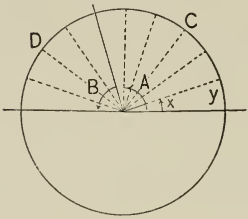
COROLLARY. *The tangent from the vertex of a pencil to a circumference is a mean proportional between the two segments of any other line of the pencil.*

In Fig. 4, $AO \cdot B_1O = A_1O \cdot BO = BO^2$. Therefore $AO : BO = BO : B_1O$.

Jun. 22

Proposition XVI.

254. Theorem. *In the same circle or in equal circles central angles are proportional to the arcs on which they stand.*



Given A and B, two central angles standing on arcs C and D, respectively.

To prove that $A : B = C : D$.

Proof. 1. If A and B are in different circles, they may be placed in the relative positions shown in the figure. § 108, def. ⊙, cor. 2

Suppose A and B divided into equal $\angle s$ x,

and suppose $A = nx$, and $B = n'x$.

(In the figure, $n = 6$, $n' = 4$.)

2. Then C is divided into n equal arcs y,
and D " " n' " " y. III, prop. I

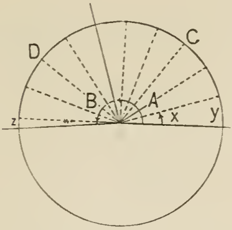
3. $$\therefore \frac{A}{B} \equiv \frac{nx}{n'x}$$

$$= \frac{n}{n'} = \frac{ny}{n'y} \equiv \frac{C}{D}.$$ Why?

$$\therefore \frac{A}{B} = \frac{C}{D}.$$ Why?

Note. The above proof assumes that A and B are commensurable, and hence that they can be divided into equal angles x. The proof on p. 181 is valid if A and B are incommensurable.

255. Proof for incommensurable case.



1. Suppose A divided into equal $\measuredangle s\ x$, and suppose $A = nx$, while $B = n'x +$ some remainder w, such that $w < x$.

   Then C is divided into n equal arcs y, and D is the sum of n' equal arcs $y +$ a remainder z, such that $z < y$.

2. Then B lies between $n'x$ and $(n' + 1)\,x$, Why?
   and D lies between $n'y$ and $(n' + 1)\,y$. Why?

3. $\therefore \dfrac{B}{A}$ and $\dfrac{D}{C}$ both lie between $\dfrac{n'}{n}$ and $\dfrac{n'+1}{n}$. Why?

   (In the figure, between $\frac{4}{6}$ and $\frac{5}{6}$.)

   And $\therefore \dfrac{B}{A}$ and $\dfrac{D}{C}$ differ by less than $\dfrac{1}{n}$. Why?

4. And $\because \dfrac{1}{n}$ can be made smaller than any assumed difference, by increasing n,

   $\therefore$ to assume any difference leads to an absurdity.

5. $\therefore \dfrac{B}{A} = \dfrac{D}{C}$; whence $\dfrac{A}{B} = \dfrac{C}{D}$.

Corollary. *In the same circle or in equal circles sectors are proportional to their angles or to their arcs.*

256. This proposition is often stated.

A central angle is measured by its intercepted arc. See § 180.

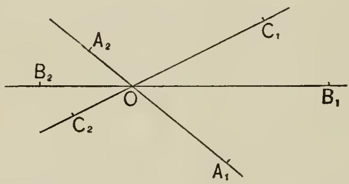
Jan 23

5. SIMILAR FIGURES.

257. Definitions. We have (§ 59) roughly defined similar figures as figures having the same shape. But this is unsatisfactory because the word *shape* is not defined. We therefore proceed scientifically to define

1. *Similar systems of points*, and then
2. *Similar figures.*

Two systems of points, A_1, B_1, C_1, ... and A_2, B_2, C_2, ..., are said to be *similar* when they can be so placed that all lines, A_1A_2, B_1B_2, C_1C_2, ..., joining corresponding points form a pencil whose vertex, O, divides each line into segments having a constant ratio r.

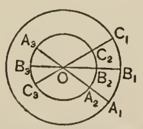


In the figure, $OA_1 : OA_2 = OB_1 : OB_2 = \cdots\cdots = r.$

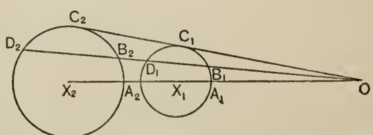
258. *Two figures* are said to be similar when their systems of points are *similar.*

The symbol of similarity $\backsim$, already mentioned, is due to Leibnitz. It is derived from the letter S.

The following are illustrations of similar figures involving circles:

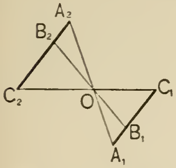


Concentric circles.

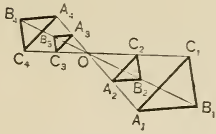


Any circles.

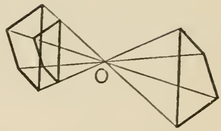
The following are illustrations of similar rectilinear figures:



Any line-segments.



Four similar triangles.



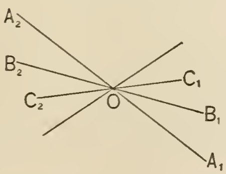
Three similar quadrilaterals.

259. When two similar figures are so placed that lines through their corresponding points form a pencil, they are said to be placed *in perspective*, and the vertex of that pencil is called their *center of similitude*.

The figures above and on p. 182 are placed in perspective, and in each case O is the center of similitude.

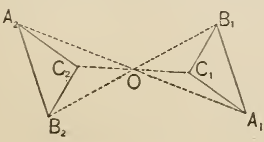
Two similar figures may evidently be so placed that the center of similitude will fall within both, or between them, or on the same side of both, as is seen in the above illustrations.

260. *Two systems of points*, A_1, B_1, C_1, ... and A_2, B_2, C_2, ..., are said to be **symmetric** *with respect to a center* O when all lines, A_1A_2, B_1B_2, C_1C_2, ..., are bisected by O.



261. Two *figures* are said to be **symmetric** *with respect to a center* when their systems of points are symmetric with respect to that center.

E.g. in the figure, △ $A_1B_1C_1$, $A_2B_2C_2$ are symmetric with respect to O.



262. In similar figures, if the ratio, r, known as the **ratio of similitude,** is 1, the figures are evidently symmetric with respect to a center. Hence Central Symmetry is a special case of Similar Figures in Perspective.

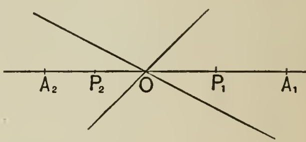
The term Center of Similitude is due to Euler.

Corollaries. 1. *Congruent figures are similar.*

For if made to coincide, any point in their plane is evidently a center of similitude, the ratio of similitude being 1. Or, they may be placed in a position of central symmetry.

2. *To any point in a system there is one and only one corresponding point of a similar system with respect to a given center.*

If A_1 and A_2 are corresponding points in two similar systems in perspective, and O is the center of similitude, then every point P_1 on OA_1 has a unique corresponding point P_2 on OA_2.

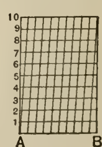


For $OA_1 : OA_2 = OP_1 : OP_2$,
$\therefore$ OP_2 is unique. § 237, cor. 1

Exercises. 392. What is the limit of $1/x$ as x increases indefinitely? of $1/(1+x)$ as $x \doteq 0$? as $x \doteq 1$?

393. In $\triangle ABC$, P is any point in AB, and Q is such a point in CA that $CQ = PB$; if PQ and BC, produced if necessary, meet at X, prove that $CA : AB = PX : QX$. (From P draw a line $\parallel AC$.)

394. In the annexed figure of a "Diagonal Scale," AB is 1 centimeter. Show how, by means of the scale and a pair of dividers, to lay off 1 millimeter, 0.5 millimeter, 0.3 millimeter, etc. On what proposition or corollary does this measurement of fractions of a millimeter depend?

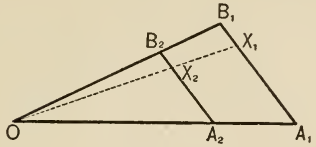


395. $ABCD$ is a parallelogram; from A a line is drawn cutting BD in E, BC in F, and DC produced in G. Prove that AE is a mean proportional between EF and EG.

396. ABC is a triangle, and through D, any point in c, DE is drawn $\parallel a$ to meet b in E; through C, CF is drawn $\parallel EB$ to meet c produced in F. Prove that AB is a mean proportional between AD and AF.

PROPOSITION XVII.

263. Theorem. *Two triangles are similar if they have two angles of the one equal to two angles of the other, respectively.*



Given the $\triangle A_1B_1C_1$, $A_2B_2C_2$, with $\angle A_1 = \angle A_2$, $\angle C_1 = \angle C_2$.

To prove that $\triangle A_1B_1C_1 \backsim \triangle A_2B_2C_2$.

Proof. 1. Place one $\triangle$ on the other so that $\angle C_1$ coincides with $\angle C_2$, as at O, OA_2 falling on OA_1. Let OX_2X_1 be *any* line through O, cutting A_2B_2 at X_2, and A_1B_1 at X_1.

Then $\because \angle A_2 = \angle A_1$,

$\therefore A_2B_2 \parallel A_1B_1$. I, prop. XVI, cor. 1

2. $\therefore OA_1 : OA_2 = OB_1 : OB_2 = OX_1 : OX_2 = \cdots\cdots = r$. Prop. X, cor. 2

3. And *all* points on OA_1 and OB_1 have their corresponding points on OA_2 and OB_2, respectively. § 262, cor. 2

4. $\therefore$ the $\triangle$ are similar, O being the center of similitude. § 258

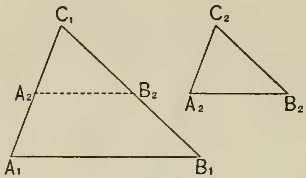
COROLLARIES. 1. *Mutually equiangular triangles are similar.*

2. *If two triangles have the sides of the one respectively parallel or perpendicular to the sides of the other, they are similar.*

For by § 86, cor. 5, they can be proved to be mutually equiangular.

Proposition XVIII.

264. Theorem. *If two triangles have one angle of the one equal to one angle of the other, and the including sides proportional, the triangles are similar.*



Given $\triangle\!\triangle\ A_1B_1C_1,\ A_2B_2C_2$, such that $\angle C_1 = \angle C_2$ and $a_1 : a_2 = b_1 : b_2$.

To prove that $\triangle A_1B_1C_1 \backsim \triangle A_2B_2C_2$.

Proof. 1. $\because \angle C_2 = \angle C_1$, $\triangle A_2B_2C_2$ can be placed on $\triangle A_1B_1C_1$ so that C_2 falls at C_1, B_2 on a_1, and A_2 on b_1.

2. Then $\because C_1A_2 : C_1A_1 = C_1B_2 : C_1B_1$,

$\therefore A_2B_2 \parallel A_1B_1$. Props. XII, V

3. $\therefore \triangle A_1B_1C_1$ and $\triangle A_2B_2C_1$ are mutually equiangular. I, prop. XVII, cor. 2

4. $\therefore \triangle A_1B_1C_1 \backsim \triangle A_2B_2C_1$ and its congruent $\triangle A_2B_2C_2$. Prop. XVII, cor. 1

Exercises. 397. ABC, DBA are two triangles with a common side AB. If P is any point on AB, and $PX \parallel AC$, and $PY \parallel AD$, meeting BC and BD in X and Y, respectively, prove that $\triangle YBX \backsim \triangle DBC$.

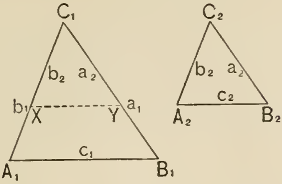
398. $ABCD$ is a quadrilateral. Prove that if the bisectors of $\angle\!\angle A, C$ meet on diagonal BD, then the bisectors of $\angle\!\angle B, D$ will meet on diagonal AC.

399. Construct a triangle, having given the base, the vertical angle, and the ratio of the remaining sides. (*Intersection of loci* and prop. XIII.)

400. In $\triangle ABC$, CM is a median; $\angle\!\angle BMC$, CMA are bisected by lines meeting a and b in R and Q, respectively. Prove that $QR \parallel AB$.

PROPOSITION XIX.

265. Theorem. *If two triangles have their sides proportional, they are similar.*



Given $\triangle\triangle\, A_1B_1C_1,\ A_2B_2C_2$ such that $a_1 : a_2 = b_1 : b_2 = c_1 : c_2$.

To prove that $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$.

Proof. 1. On C_1A_1, C_1B_1 lay off $C_1X = b_2$, and $C_1Y = a_2$, and draw XY.

Then $\because a_1 : a_2 = b_1 : b_2$, and $\angle C_1 \equiv \angle C_1$,

2. $\therefore \triangle XYC_1 \sim \triangle A_1B_1C_1$. Prop. XVIII

3. But $a_1 : a_2 = c_1 : c_2$, Given

and $a_1 : a_2 = c_1 : XY$. Prop. X, cor. 3

4. $\therefore c_1 : c_2 = c_1 : XY$, Why?

and $c_2 = XY$. § 237, cor. 2

5. $\therefore \triangle XYC_1 \cong \triangle A_2B_2C_2$, I, prop. XII

and $\therefore \triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$. Steps 2, 5

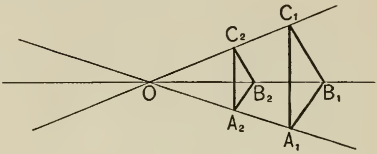
Exercises. 401. The product of the two segments of any chord drawn through a given point within a circle equals the square of half the shortest chord that can be drawn through that point.

402. If P is a point on AB produced, the tangents from P to all circumferences through A and B are equal, and hence such points are concyclic.

403. If from any point P on the side CA of a right-angled triangle ABC, PQ is drawn perpendicular to the hypotenuse AB at Q, then $AP \cdot AC = AQ \cdot AB$. Suppose P to be taken (1) at C; (2) at A; (3) on AC produced.

Proposition XX.

266. Theorem. *Similar triangles have their corresponding sides proportional and their corresponding angles equal.*



Given two similar triangles, $A_1B_1C_1$, $A_2B_2C_2$, A_1 corresponding to A_2, B_1 to B_2, C_1 to C_2.

To prove that $A_1B_1 : A_2B_2 = B_1C_1 : B_2C_2 = \ldots\ldots$ and that $\angle B_1 = \angle B_2 \ldots\ldots$

Proof. 1. Suppose the $\triangle$ placed in perspective.
Then $OA_1 : OA_2 = OB_1 : OB_2 = OC_1 : OC_2$. § 258

2. $\therefore A_1B_1 \parallel A_2B_2$, and so for other sides. Props. XII, V

3. $\therefore \angle OB_1A_1 = \angle OB_2A_2$, and $\angle C_1B_1O = \angle C_2B_2O$. I, prop. XVII, cor. 2

4. $\therefore \angle B_1 = \angle B_2$, and so for other angles. Ax. 2

5. Also, $OB_1 : OB_2 = A_1B_1 : A_2B_2$,
$= B_1C_1 : B_2C_2$. Prop. X, cor. 3

6. $\therefore A_1B_1 : A_2B_2 = B_1C_1 : B_2C_2$, and so for other sides.

Note. This is the converse of props. XVII, XIX.

Corollaries. 1. *The corresponding altitudes of two similar triangles have the same ratio as any two corresponding sides.*

Why?

2. *The corresponding sides of similar triangles are opposite the equal angles.*

In what step is this proved?

267. Summary of Propositions concerning Similar Triangles.

Two triangles are similar if

1. (*a*) *Two angles of the one equal two angles of the other.* Prop. XVII

   (*b*) *They are mutually equiangular.* Prop. XVII, cor. 1

   (*c*) *The sides of the one are parallel to the sides of the other.* Prop. XVII, cor. 2

   (*d*) *The sides of the one are perpendicular to the sides of the other.* Prop. XVII, cor. 2

2. *One angle of the one equals one angle of the other and the including sides are proportional.* Prop. XVIII

3. *Their corresponding sides are proportional.* Prop. XIX

If two triangles are similar,

1. *They are mutually equiangular.* Prop. XX

2. *Their corresponding sides are proportional.* Prop. XX

3. *Their corresponding altitudes are proportional to their corresponding sides.* Prop. XX, cor. 1

268. It should further be observed that, in general,

Three conditions determine congruence. (See § 90.)
Two conditions determine similarity.

For these conditions are

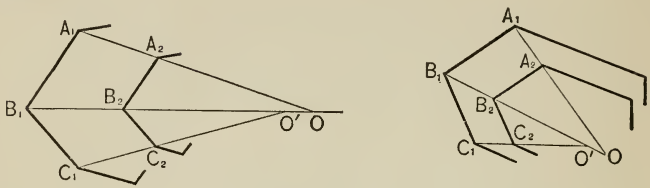
1. Two angles equal. (Prop. XVII.)
2. One angle and one ratio. (Prop. XVIII.)
3. Two ratios; for if the sides are a, b, c, and a', b', c', then if $\frac{a}{b}=\frac{a'}{b'}$, and $\frac{a}{c}=\frac{a'}{c'}$, the $\triangle$ are similar, since $\frac{b}{c}$ must also equal $\frac{b'}{c'}$.

Exercises. 404. If X is any point in the side a, or a produced, of $\triangle ABC$, and if r_b and r_c are the radii of circles circumscribed about $\triangle ABX$ and $\triangle AXC$, respectively, then $r_b : r_c = c : b$. (Join the centers and prove two triangles similar.)

405. If one of the parallel sides of a trapezoid is double the other, prove that the diagonals intersect one another in a point of trisection.

Proposition XXI.

269. Theorem. *If two polygons are mutually equiangular and have their corresponding sides proportional, they are similar.*



Given two polygons, $A_1B_1C_1$ ..... and $A_2B_2C_2$ ....., such that

$$\angle A_1 = \angle A_2,\ \angle B_1 = \angle B_2, \ldots\ldots,$$

and $$A_1B_1 : A_2B_2 = B_1C_1 : B_2C_2 = \ldots\ldots$$

To prove that $$A_1B_1C_1 \ldots\ldots \backsim A_2B_2C_2 \ldots\ldots$$

Proof. 1. Place $A_2B_2 \parallel A_1B_1$. Then $\because$ the $\angle$s of one polygon $=$ the corresponding $\angle$s of the other, the remaining sides may be made parallel respectively.
I, prop. XVII, cor. 5

2. If $A_1B_1 > A_2B_2$, then $B_1C_1 > B_2C_2$, ..., because the ratios are equal.

3. Draw A_1A_2, B_1B_2, ... Then $A_1B_1B_2A_2$ is not a $\square$; also $B_1C_1C_2B_2$, etc.; and A_1A_2 meets B_1B_2 as at O, B_1B_2 meets C_1C_2 as at O', etc. I, prop. XXIV

4. But $B_1O' : B_2O' = B_1C_1 : B_2C_2$
$$= A_1B_1 : A_2B_2 = B_1O : B_2O,$$
Prop. X, cor. 3

which is impossible unless O and O' coincide.
Prop. XI, cor.

5. $\therefore$ the two figures are similar, and O is the center of similitude. § 258

In step 2, if $A_1B_1 = A_2B_2$, then $B_1C_1 = B_2C_2$, ..., and the polygons are congruent and therefore similar. § 262, cor. 1

COROLLARIES. 1. *If two polygons are similar, they are mutually equiangular and their corresponding sides are proportional.*

For if placed in perspective as on p. 190,

1. $OA_1 : OA_2 = OB_1 : OB_2$. § 258
2. $\therefore A_1B_1 \parallel A_2B_2$, and so for other sides. Prop. XII
3. $\therefore \angle B_1 = \angle B_2$, and so for other angles. I, prop. XVII, cor. 5
4. Also $A_1B_1 : A_2B_2 = B_1O : B_2O = B_1C_1 : B_2C_2 = ...$ Prop. X, cor. 3

2. *Polygons similar to the same polygon are similar to each other.*

For they have angles equal to those of the third polygon, and the ratios of their sides equal the ratios of the sides of the third polygon.

3. *The perimeters of similar polygons have the same ratio as the corresponding sides.*

For by cor. 1, $A_1B_1 : A_2B_2 = B_1C_1 : B_2C_2 = ... = r$. $\therefore A_1B_1 + B_1C_1 + ... : A_2B_2 + B_2C_2 + ... = r$. (Why?)

4. *Two similar polygons can be divided into the same number of triangles similar each to each, and similarly placed.*

For O and O' coincide, and the figures can be placed having O within each. The triangles A_1OB_1, A_2OB_2 are then similar, by prop. XVII.

Exercises. 406. If from a point outside a circle a pair of tangents and a secant are drawn, the quadrilateral formed by joining in succession the four points thus determined on the circumference has the rectangles of its opposite sides equal.

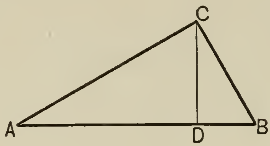
407. AB is a diameter, and from A a line is drawn to cut the circumference in C and the tangent from B in D. Prove that the diameter is the mean proportional between AC and AD.

408. In $\square$ $ABCD$, P, Q are points in a line parallel to AB; PA and QB meet at R, and PD and QC meet at S. Prove that $RS \parallel AD$.

409. Chords AB, CD are produced to meet at P, and PF is drawn parallel to DA to meet CB produced in F. Prove that PF is the mean proportional between FB and FC.

Proposition XXII.

270. Theorem. *In a right-angled triangle the perpendicular from the vertex of the right angle to the hypotenuse divides the triangle into two triangles which are similar to the whole and to each other.*



Given $\triangle ABC$, with $\angle C$ a right angle, and $CD \perp AB$.

To prove that (1) $\triangle ACD \backsim \triangle ABC$.

(2) $\triangle CBD \backsim \triangle ABC$.

(3) $\triangle ACD \backsim \triangle CBD$.

Proof. 1. $\because \angle CDA = \angle ACB$, Why?

and $\angle A \equiv \angle A$,

$\therefore \triangle ACD \backsim \triangle ABC$, which proves (1). Prop. XVII

2. Similarly $\triangle CBD \backsim \triangle ABC$, which proves (2). Prop. XVII

3. $\therefore \triangle ACD \backsim \triangle CBD$, which proves (3). Prop. XXI, cor. 2

Corollaries. 1. *Either side of a right-angled triangle is the mean proportional between the hypotenuse and its segment adjacent to that side.*

For from step 1, $AB : AC = AC : AD$; and from 2, $AB : BC = BC : DB$.

2. *The perpendicular from the vertex of the right angle to the hypotenuse is the mean proportional between the segments of the hypotenuse.*

For from step 3, $AD : CD = CD : DB$.

EXERCISES.

410. Prove the converse of prop. XXII: If the perpendicular drawn from the vertex of a triangle to the base is the mean proportional between the segments of the base, the triangle is right-angled.

411. Prove that any chord of a circle is the mean proportional between its projection on the diameter from one of its extremities, and the diameter itself.

412. In the figure on p. 192, if AD represents three units, and DB represents one unit, what number is represented by CD?

413. Prove that if a perpendicular is let fall from any point on a circumference, to any diameter, it is the mean proportional between the segments into which it divides that diameter.

414. Prove that if two fixed parallel tangents are cut by a variable tangent, the rectangle of the segments of the latter is constant.

415. Through any point in the common chord of two intersecting circumferences two chords are drawn, one in each circle. Prove that the four extremities of these chords are concyclic.

416. If the bisectors of the interior and exterior angles at B, in the figure of prop. XXII, meet b at F and E, respectively, prove that BC is the mean proportional between FC and CE.

417. Calculate each of the segments into which the bisectors of the angles of a triangle divide the opposite sides, the lengths of the sides being 9 in., 12 in., and 15 in., respectively.

418. From the points A, B, on a line AB, 25 in. long, perpendiculars AC, BD are erected such that $AC = 13$ in., $BD = 7$ in. On AB the point O is taken such that $\angle BOD = \angle COA$. Calculate the distances AO, OB.

419. Given a trapezoid $ABCD$, with the non-parallel sides AD, BC divided at E, F, respectively, in the ratio of 2 to 3, to calculate the length of EF, knowing that $AB = 12.45$ in., and $DC = 38.5$ in.

420. Calculate the sides of a right triangle, knowing that their respective projections on the hypotenuse are 2.88 in. and 5.12 in.

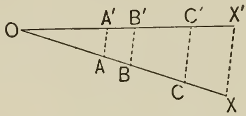
421. The two sides of a right triangle are respectively 10 in. and 24 in. Required the lengths of their projections on the hypotenuse, and the distance of the vertex of the right angle from the hypotenuse. (To 0.001.)

422. The two sides of a right triangle are respectively 3.128 in. and 4.275 in. Required the lengths of the two segments into which the bisector of the right angle divides the hypotenuse. (To 0.001.)

6. PROBLEMS.

PROPOSITION XXIII.

271. Problem. *To divide a line-segment into parts proportional to the segments of a given line.*



Given the line OX', and the line OX divided into segments OA, AB, ...

Required to divide OX' into segments proportional to OA, AB, ...

Construction. 1. Placing the lines oblique to each other at a common end-point O, draw XX'. § 28

2. From A, B, ... draw lines $\parallel XX'$, cutting OX' at A', B', ... I, prop. XXXIII

Then OX' is divided as required.

Proof. $\because$ OX, OX' are two transversals of a pencil of $\parallel$'s, the corresponding segments are in proportion. Prop. X

COROLLARIES. 1. *A given line can be divided into parts proportional to any number of given lines.*

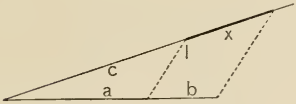
For that number of given lines may be laid off as OA, AB, BC, ... on OX.

2. *A line can be divided into any number of equal parts.*

NOTE. While a straight line can be divided into any number of equal parts, by means of the straight edge and the compasses, a circumference cannot be divided into 7, 9, 11, 13, and, in general, any prime number of equal parts beyond 5. The exceptions are noted in Book V.

PROPOSITION XXIV.

272. Problem. *To find the fourth proportional to three given lines.*



Given three lines, a, b, c.

Required to find x such that $a : b = c : x$.

Construction. 1. From the vertex of a pencil of two lines, with the compasses lay off a, b, in order, on one line, and c on the other line.

2. Join the end-points of a, c, remote from the vertex, by l. § 28

3. From the end-point of b, remote from a, draw a line parallel to l. I, prop. XXXIII

This will cut off x, the line required.

Proof. $a : b = c : x$. Prop. X, cor. 1

273. Definition. If $a : b = b : x$, x is called the *third proportional* to a and b.

COROLLARY. *The third proportional to two given lines can be found.*

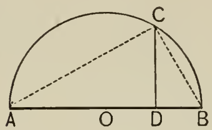
For to find x such that $a : b = b : x$, make $c = b$ in the above solution.

Exercises. 423. The problem admits of a considerable variation of the figure, as suggested by the figure given in ex. 383. Invent another solution from this suggestion.

424. How many inches in the fourth proportional to lines respectively 2 in., 3 in., 5 in. long? In the third proportional to lines respectively 2 in., 7 in. long?

Proposition XXV.

274. Problem. *To find the mean proportional between two given lines.*



Given two lines, AD, DB.

Required to find the mean proportional between them.

Construction. 1. Placing AD, DB end to end in the same line, bisect AB at O. I, prop. XXXI

2. With center O and radius OB, describe a circle. § 109

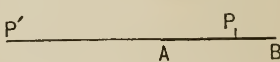
3. From D draw $DC \perp AB$, to meet circumference at C. I, prop. XXIX

Then CD is the mean proportional.

Proof. $AD : CD = CD : DB$. Prop. XXII, cor. 2, and § 238

275. Definition. A line is said to be divided in **extreme and mean ratio** by a point when one of the segments is the mean proportional between the whole line and the other segment.

Thus, AB is divided internally in extreme and mean ratio at P, if $AB : AP = AP : PB$; and externally in such ratio at P', if $AB : AP' = AP' : P'B$.



To say that $AB : AP = AP : PB$ is merely to say that $AP^2 = AB \cdot PB$. This division is often known as the **Golden Section** or the **Median Section**.

If the student understands quadratic equations he will see that if the length of AB is 6, and if $AP = x$, then $PB = 6 - x$, and

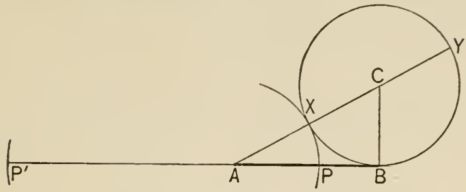
$$\because AP^2 = AB \cdot PB,$$

$$\therefore x^2 = 6(6 - x),$$

or $x^2 + 6x - 36 = 0$. Solving, $x = -3 \pm 3\sqrt{5}$.

PROPOSITION XXVI.

276. Problem. *To divide a line in extreme and mean ratio.*



Given the line AB.

Required to divide AB in extreme and mean ratio; *i.e.* to find P such that $AB \cdot PB = AP^2$.

Construction. 1. Draw $CB \perp AB$ and $= \frac{1}{2} AB$.

2. Describe a ⊙ with center C and radius CB.

3. Draw AC cutting the circumference in X and Y.

4. Describe two arcs with center A and radii AX and AY, thus fixing points P, P'.

These are the required points.

Proof for point P.

$$\begin{aligned} AB^2 &= AX \cdot AY \\ &= AP\,(AX + XY) \\ &= AP\,(AP + AB) \\ &= AP^2 + AP \cdot AB. \end{aligned}$$

$$\therefore AB\,(AB - AP) = AP^2.$$

$$\therefore AB \cdot PB = AP^2.$$

Proof for point P'.

$$\begin{aligned} AB^2 &= AY \cdot AX \\ &= P'A\,(AY - XY) \\ &= P'A\,(P'A - AB) \\ &= P'A^2 - AB \cdot P'A. \end{aligned}$$

$$\therefore AB\,(AB + P'A) = P'A^2.$$

$$\therefore AB \cdot P'B = P'A^2.$$

$\therefore AB$ is divided internally at P and externally at P' in Golden Section.

It should be noticed that if the sense of the lines as positive or negative is considered (that is, considering $AP = - PA$), the above solutions would be identical if X and Y were interchanged, and P' substituted for P.

Proposition XXVII.

277. Problem. *On a given line-segment as a side corresponding to a given side of a given polygon, to construct a polygon similar to that polygon.*

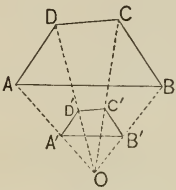


Fig. 1.

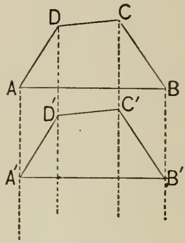


Fig. 2.

Given the polygon $ABCD$ and the line-segment $A'B'$.

Required to construct on $A'B'$ as a side corresponding to AB, a polygon $A'B'C'D' \backsim ABCD$.

Construction. 1. In Fig. 1, place $A'B' \parallel AB$. I, prop. XXXIII

2. Draw AA', BB', meeting at O; draw OC, OD. § 28

3. Draw $B'C' \parallel BC$, $C'D' \parallel CD$. I, prop. XXXIII

4. Draw $D'A'$. Then $A'B'C'D' \backsim ABCD$.

Proof. 1. $\because OA : OA' = OB : OB' = OC : OC' = OD : OD'$, § 244

$\therefore D'A' \parallel DA$. Prop. XII

2. $\because A'B' : AB = OB' : OB = B'C' : BC = \cdots\cdots,$

Prop. X, cor. 3

and $\angle C'B'A' = \angle CBA$, and so for the other $\angle$s,

I, prop. XVII, cor. 2; ax. 3

$\therefore A'B'C'D' \backsim ABCD$. Prop. XXI

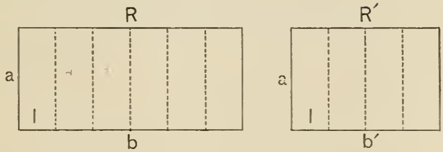
If $A'B' = AB$, as in Fig. 2, draw from C, D $\parallel$'s to AA'; otherwise the construction is as above. It is left to the student to prove $D'A' \parallel DA$, and $A'B'C'D' \cong ABCD$. $\therefore A'B'C'D' \backsim ABCD$ by § 262, cor. 1.

BOOK V. — MENSURATION OF PLANE FIGURES. REGULAR POLYGONS AND THE CIRCLE.

1. MENSURATION OF PLANE FIGURES.

Proposition I.

278. Theorem. *Two rectangles having equal altitudes are proportional to their bases.*



Given two rectangles R and R', with altitude a, and with bases b, b', respectively.

To prove that $R : R' = b : b'$.

Proof. 1. Suppose b and b' divided into equal segments, l, and suppose $b = nl$, and $b' = n'l$.

(In the figures, $n = 6$, $n' = 4$.)

Then if ⊥s are erected from the points of division,

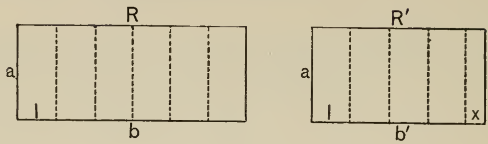
$R = n$ congruent rectangles al,

and $R' = n'$ " " "

2. $\therefore \dfrac{R}{R'} = \dfrac{n \cdot al}{n' \cdot al} = \dfrac{n}{n'} = \dfrac{b}{b'}$. Why?

Note. The above proof assumes that b and b' are commensurable, and hence that they can be divided into equal segments l. The proposition is, however, entirely general. The proof on p. 200 is valid if b and b' are incommensurable.

279. Proof for incommensurable case.



1. Suppose b divided into equal segments l,

and suppose $b = nl$,

while $b' = n'l +$ some remainder x,

such that $x < l$.

Then if $\perp$'s are erected from the points of division,

$R = n$ congruent rectangles al,

and $R' = n'$ " " $al +$ a remainder ax,

such that $ax < al$.

2. Then b' lies between $n'l$ and $(n'+1)\,l$, Why?

(In the figure, between $4\,l$ and $5\,l$.)

and R' lies between $n' \cdot al$ and $(n'+1) \cdot al$. Why?

3. $\therefore \dfrac{b'}{b}$ and $\dfrac{R'}{R}$ both lie between $\dfrac{n'}{n}$ and $\dfrac{n'+1}{n}$, Why?

(In the figure, between $\frac{4}{6}$ and $\frac{5}{6}$.)

and $\therefore$ they differ by less than $\dfrac{1}{n}$. Why?

(In the figure, by less than $\frac{1}{6}$.)

4. And $\because \dfrac{1}{n}$ can be made smaller than any assumed difference, by increasing n,

$\therefore$ to assume any difference leads to an absurdity.

5. $$\therefore \frac{b'}{b} = \frac{R'}{R}, \text{ whence } \frac{R}{R'} = \frac{b}{b'}.$$

Note. The proof will be noticed to be essentially that of pp. 171, 181.

COROLLARIES. 1. *Rectangles having equal bases are proportional to their altitudes.*

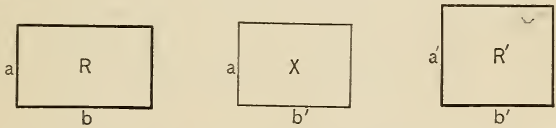
For they can be turned through 90° so as to interchange base and altitude.

2. *Triangles having equal altitudes are proportional to their bases; having equal bases, to their altitudes.* (Why?)

3. *Parallelograms having equal bases are proportional to their altitudes; having equal altitudes, to their bases.*

PROPOSITION II.

280. Theorem. *Two rectangles have the same ratio as the products of (the numerical measures of) their bases and altitudes.*



Given two rectangles R, R', with bases b, b', and altitudes a, a', respectively.

To prove that $R : R' = ab : a'b'$.

Proof. 1. Let X be a rectangle of altitude a and base b'.

Then $$\frac{R}{X} = \frac{b}{b'},$$ Prop. I

and $$\frac{X}{R'} = \frac{a}{a'}.$$ Prop. I, cor. 1

2. $$\therefore \frac{R}{R'} = \frac{ab}{a'b'},$$ by multiplying corresponding members of the two equations. Ax. 6

NOTE. Thus again appears the relation between geometry and algebra set forth in § 221, that to the product of two numbers corresponds the rectangle of two lines.

281. Definition. To **measure** a surface is to find its ratio to some unit. The unit of measure, multiplied by this ratio, is called the *area.*

Thus, in a surface 4 ft. long by 2 ft. broad, the ratio of the surface to 1 sq. ft. is 8, and 8 sq. ft. is the area.

Corollaries. 1. *Parallelograms (or triangles) have the same ratio as the products of their bases and altitudes.* (Why?)

For a parallelogram equals a rectangle of the same base and the same altitude, II, prop. I, cor. 1. See also (for the triangle) II, prop. II, cor. 1.

2. *The area of a rectangle equals the product of its base and altitude.*

That is, the number which represents its square units of area is the product of the two numbers which represent its base and altitude.

For in prop. II, if $R' = 1$, the square unit of area, then a' and b' must each equal 1, the unit of length. Hence $R/1 = ab/1$, or $R = ab$.

3. *The area of a parallelogram equals the product of its base and altitude; of a triangle, half that product.*

See the proof under cor. 1.

4. *The area of a square equals the second power of its side.*

This is the reason that the *second power* of a *number* is called its *square.*

5. *The area of a trapezoid equals the product of its altitude and half the sum of its bases.* (Why?)

See II, prop. III.

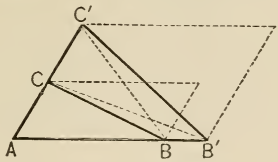
Exercises. 425. Prove that any quadrilateral is divided by its interior diagonals into four triangles which form a proportion.

426. ABC is a triangle, and P is any point in BC; from P are drawn two parallels to CA, BA, meeting AB, AC in X, Y, respectively. Prove that $\triangle AXY$ is a mean proportional between $\triangle BPX$ and $\triangle PCY$. Investigate when P is on CB produced.

427. Suppose D, E, the mid-points of sides b, a of $\triangle ABC$, to be joined; draw AE and BD, intersecting at O. Prove that $\triangle BEO$ is a mean proportional between $\triangle$s DOE and ABO. Investigate when $DE \parallel AB$, but D and E are not mid-points of b, a.

PROPOSITION III.

282. Theorem. *Triangles, or parallelograms, which have an angle in one equal to an angle in the other, have the same ratio as the products of the including sides.*



Given two triangles ABC, $AB'C'$, having an angle, A, of one equal to an angle, A, of the other.

To prove that $\triangle ABC : \triangle AB'C' = AB \cdot AC : AB' \cdot AC'$.

Proof. 1. Suppose them placed with $\angle A$ in common; draw BC', $B'C$.

Then $\triangle ABC : \triangle AB'C = AB : AB'$. Why?

2. And $\triangle AB'C : \triangle AB'C' = AC : AC'$,

their bases being AC, AC'. Why?

3. $\therefore \triangle ABC : \triangle AB'C' = AB \cdot AC : AB' \cdot AC'$. Ax. 6

4. And $\because$ the ▱s in the figure are double the △s, the theorem is true for parallelograms.

COROLLARY. *Similar triangles have the same ratio as the squares of their corresponding sides.*

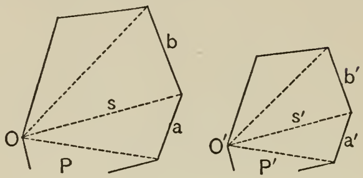
For if the △s are similar, $BC \parallel B'C'$, and the ratio $AB : AB'$ equals, and may be substituted for, the ratio $AC : AC'$, thus making the second member of step 3, $AB^2 : AB'^2$.

Exercises. 428. Prove prop. III, changed to read, "an angle in one *supplemental* to an angle in the other."

429. Prove the converse of prop. III, cor.: If two triangles have the same ratio as the squares of any two corresponding sides, they are similar.

Proposition IV.

283. Theorem. *Similar polygons have the same ratio as the squares of their corresponding sides.*



Given P and P', two similar polygons; sides a, b, ..., corresponding to sides a', b', ...; diagonal s corresponding to diagonal s'.

To prove that $$P : P' = a^2 : a'^2.$$

Proof. 1. Suppose P and P' divided into similar △s of bases a and a', b and b', ..., by diagonals from corresponding points O, O'. IV, prop. XXI, cor. 4

Then $\triangle\, Oa : \triangle\, O'a' = a^2 : a'^2$, Prop. III, cor.

and $\triangle\, Oa : \triangle\, O'a' = s^2 : s'^2 = \triangle\, Ob : \triangle\, O'b' = \cdots\cdots$ Why?

2. $\therefore$ △s $Oa + Ob + \cdots\cdots :$ △s $O'a' + O'b' + \cdots\cdots = \triangle\, Oa : \triangle\, O'a'$. IV, prop. VI

3. $\because$ △s $Oa + Ob + \cdots\cdots = P$, and △s $O'a' + O'b' + \cdots\cdots = P'$,

$$\therefore P : P' = a^2 : a'^2.$$

Exercises. 430. If the vertices, A, B, C, of a triangle are joined to a point O within the triangle, and if AO produced cuts a at D, then

$$\triangle\, ABO : \triangle\, AOC = BD : DC.$$

431. If two triangles are on equal bases and between the same parallels, then any line parallel to their bases, cutting the triangles, will cut off equal triangles.

432. Two equilateral triangles have their areas in the ratio of 1 : 2. Find the ratio of their sides to the nearest 0.01.

2. PARTITION OF THE PERIGON.

PROPOSITION V.

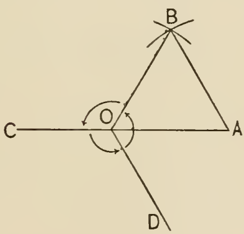
284. Problem. *To bisect a perigon.*

Construction and Proof. A special case under I, prop. XXVIII.

COROLLARY. *A perigon can be divided into 2^n equal angles.*

PROPOSITION VI.

285. Problem. *To trisect a perigon.*



Given the perigon with vertex O.

Required to trisect it.

Construction. 1. On any line OA, from O, construct an equilateral $\triangle OAB$. Authority?

2. Produce AO to C, and bisect $\angle COB$ by OD.

Then the perigon is trisected by OB, OC, OD.

Proof. 1. $\because \angle AOB = 60°$, I, prop. XIX, cor. 8

$\therefore \angle BOC$, supplement of $\angle AOB = 120°$.

2. $\therefore \angle COB$, conjugate of $\angle BOC = 240°$.

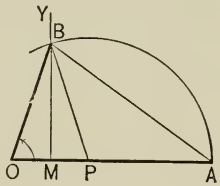
3. $\therefore \angle COD = \angle DOB = 120°$. Const. 2

Corollary. *A perigon can be divided into $3 \cdot 2^n$ equal angles.*

For if $n = 0$, then $3 \cdot 2^n = 3 \cdot 1 = 3$, so that the corollary reduces to the problem itself. If $n = 1$, then $3 \cdot 2^n = 6$, and by bisecting $\measuredangle s$ BOC, COD, DOB, the perigon is divided into 6 equal angles. Similarly, by bisecting again, the perigon is divided into $3 \cdot 2^2 = 12$ equal angles, and so on.

Proposition VII.

286. Problem. *To divide a perigon into five equal angles.*



Given the perigon with vertex O.

Required to divide it into five equal angles.

Construction. 1. Draw OA, and divide it at P so that $OP \cdot OA = PA^2$. IV, prop. XXVI

2. Draw MY, the $\perp$ bisector of OP. I, prop. XXXI

3. With center P and radius PA describe an arc cutting MY in B. § 109

4. Draw OB.

Then $\angle AOB = \frac{1}{5}$ of a perigon.

Proof. 1. Draw AB and PB.

Then $\because OP \cdot OA = PA^2$,

and $\because OA > PA$,

$\therefore OP < PA$, and $\therefore MP < PA$.

$\therefore \widehat{AB}$ cuts MY.

2. $\because \triangle OPB$ is isosceles, $OB = PB = PA$, the radius.

3. Also, $OA^2 + OB^2 = AB^2 + 2\,OM \cdot OA$.

II, prop. IX, cor. 1

And $\because 2\,OM = OP$,

$\therefore OA^2 + OB^2 = AB^2 + OP \cdot OA$.

4. And $\because OB^2 = PA^2 = OP \cdot OA$,

$\therefore OA^2 + OB^2 = AB^2 + OB^2$, from step 3,

$\therefore OA^2 = AB^2$,

and $\therefore OA = AB$.

5. $\therefore \angle OBA = \angle O = \angle BPO$, I, prop. III

$= \angle A + \angle PBA$. I, prop. XIX

And $\because \angle A = \angle PBA$, I, prop. III

$\therefore \angle O = 2 \angle A$.

6. $\therefore \angle O$ is $\frac{2}{5}$ of a st. $\angle$, or $\frac{1}{5}$ of a perigon.

I, prop. XIX

COROLLARY. *A perigon can be divided into $5 \cdot 2^n$ equal parts.*

For if $n = 0$, then $5 \cdot 2^n = 5 \cdot 1 = 5$, so that the corollary reduces to the problem itself. If $n = 1$, then $5 \cdot 2^n = 5 \cdot 2 = 10$, and by bisecting $\angle AOB$, the resulting angle is $\frac{1}{10}$ of a perigon. Similarly, by bisecting again, $\frac{1}{20}$ of a perigon is formed, and so on.

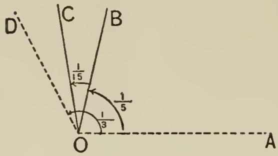
Exercises. **433.** In the figure on p. 206, let $OP = x$, $PA = r$; then show that $x = \frac{r}{2}(\sqrt{5} - 1)$. (Omit exs. 433, 434 if the student has not had quadratic equations.)

434. In the same figure, if $OP = x$ and $OA = a$, show that $x = \frac{a}{2}(3 - \sqrt{5})$.

435. On the sides a, b, c, of an equilateral triangle, points X, Y, Z are so taken that $BX : XC = CY : YA = AZ : ZB = 2 : 1$. Find the ratio of $\triangle XYZ$ to $\triangle ABC$.

Proposition VIII.

287. Problem. *To divide a perigon into fifteen equal angles.*



Solution. 1. Make $\angle AOB = \frac{1}{5}$ of a perigon. Prop. VII

2. Make $\angle AOD = \frac{1}{3}$ of a perigon. Prop. VI

3. Bisect $\angle BOD$. I, prop. XXVIII

4. Then $\angle BOC = \frac{1}{2}(\frac{1}{3} - \frac{1}{5})$ perigon $= \frac{1}{15}$ of a perigon.

Corollary. *A perigon can be divided into $15 \cdot 2^n$ equal angles.*

Explain.

288. Note. That a perigon could be divided into 2^n, $3 \cdot 2^n$, $5 \cdot 2^n$, $15 \cdot 2^n$ equal angles, was known as early as Euclid's time. By the use of the compasses and straight edge no other partitions were deemed possible. In 1796 Gauss found, and published the fact in 1801, that a perigon could be divided into 17, and hence into $17 \cdot 2^n$ equal angles; furthermore, that it could be divided into $2^m + 1$ equal angles if $2^m + 1$ was a prime number; and, in general, that it could be divided into a number of equal angles represented by the product of different prime numbers of the form $2^m + 1$. Hence it follows that a perigon can be divided into a number of equal angles represented by the product of 2^n and one or more different prime numbers of the form $2^m + 1$. It is shown in the Theory of Numbers that if $2^m + 1$ is prime, m must equal 2^p; hence the general form for the prime numbers mentioned is $2^{2^p} + 1$. Gauss's proof is only semi-geometric, and is not adapted to elementary geometry.

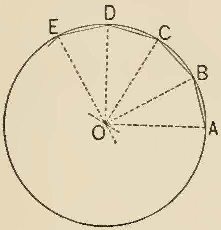
Exercises. 436. Including the divisions of a perigon suggested by Gauss, there are 25 possible divisions below 100. What are they?

437. As in ex. 436, there are 13 possible divisions between 100 and 300. What are they?

3. REGULAR POLYGONS.

PROPOSITION IX.

289. Problem. *To inscribe in a circle a regular polygon having a given number of sides.*



Given a circle with center O and radius OA.

Required to inscribe in the circle a regular n-gon.

Construction. 1. Divide the perigon O into n equal parts (n being limited as in props. V–VIII and cors.) as AOB, BOC, COD, ..., B, C, D, ... lying on the circumference.

2. Draw AB, BC, CD ..... § 28

Then $ABCD$ ..... is an inscribed regular n-gon.

Proof. 1. $\triangle AOB \cong \triangle BOC \cong \triangle COD \cong \cdots\cdots$,

and $AB = BC = CD = \cdots\cdots$ I, prop. I

2. $\therefore \widehat{AB} = \widehat{BC} = \widehat{CD} = \cdots\cdots$ III, prop. IV

3. $\therefore \angle DCB = \angle CBA = \cdots\cdots$,

$\because$ each stands on $(n-2)$ arcs equal to $\widehat{AB}$.

III, prop. XI, cor. 1

4. $\therefore ABCD$ ..... is an inscribed regular polygon.

§§ 92, 201

Corollaries. 1. *The side of an inscribed regular hexagon equals the radius of the circle.*

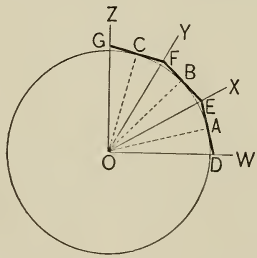
Then $\angle AOB = \frac{1}{6}$ of $360° = 60°$; $\therefore \angle BAO$, which $= \angle OBA = 60°$. $\therefore \triangle ABO$ is equilateral.

2. *An inscribed equilateral polygon is regular.*

For by step 3 of the proof it is also equiangular; and being both equilateral and equiangular, it is regular.

Proposition X.

290. Problem. *To circumscribe about a circle a regular polygon having a given number of sides.*



Given a circle with center O and radius OA.

Required to circumscribe about this circle a regular n-gon.

Construction. 1. Divide the perigon O into n equal parts (n being limited as in props. V–VIII and cors.) by lines OW, OX, OY, ...

2. Bisect $\angle$s WOX, XOY, ... by radii to A, B, ... I, prop. XXVIII

3. From A, B, C, ... draw tangents to meet OW at D, OX at E, ... III, prop. XXVI

Then $DEFG$ ..... is the required polygon.

Proof. 1. $\because DE \perp OA$, $EF \perp OB$, ..., III. prop. IX, cor. 2

$\therefore \triangle OAE \cong \triangle OBE$, and $AE = BE$, $OE = OE$. I, prop. II

2. $\therefore$ the tangents from A and B meet OX at the same point, E.

3. And $\because \angle DOE = \angle EOF$, Const. 1

and $\angle OED = \angle FEO$, Step 1

$\therefore \triangle DOE \cong \triangle EOF$, and $DE = EF$. Why ?

4. Also, $\because \angle GFE = \angle FED$, each being the supplement of an $\angle$ equal to $\angle WOX$, (Name the $\angle$s.) I, prop. XXI, cor.

$\therefore DEFG$ ..... is a circumscribed regular polygon. §§ 92, 201

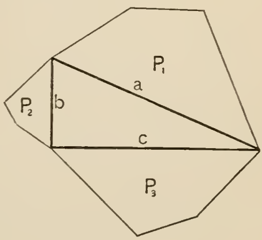
COROLLARIES. 1. *The side of a regular hexagon circumscribed about a circle of diameter* 1, *is* $1/\sqrt{3}$, *or* $\frac{1}{3}\sqrt{3}$.

For it is (as in prop. IX, cor. 1) the side of an equilateral $\triangle$ whose altitude is $\frac{1}{2}$. This is easily shown to be $1/\sqrt{3}$. (Show it.)

2. *A circumscribed equiangular polygon is regular.*

Prove that any two adjacent sides are equal.

Exercises. 438. In a right-angled triangle, any polygon on the hypotenuse equals the sum of two similar polygons described on the sides as corresponding sides of those polygons.



(Suggestion : $P_2 : P_3 = b^2 : c^2$;

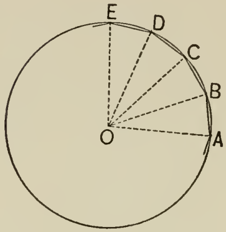
$\therefore P_2 + P_3 : P_3 = b^2 + c^2 : c^2$
$= a^2 : c^2 = P_1 : P_3$;

$\therefore P_2 + P_3 : P_1 = P_3 : P_3 = 1$. This is one of the generalized forms of the Pythagorean theorem.)

439. If r is the radius of the circle, and s is the side of the inscribed equilateral triangle, then $s = r\sqrt{3}$.

Proposition XI.

291. Problem. *To circumscribe a circle about a given regular polygon.*



Given the regular polygon $ABCD$ .....

Required to circumscribe a circle about it.

Construction. Bisect $\angle$s DCB, CBA, the bisectors meeting at O.
Then O is the center and OB the radius.

Proof. 1. Draw OA, OD, OE, ... Then $\because$ $\angle$s OCB, CBO are halves of oblique $\angle$s, each is less than a rt. $\angle$.

2. $\therefore$ CO and BO cannot be $\parallel$, and they meet as at O.

3. And $\because \angle CBO = \angle OBA$, Const.

and $AB = BC$, § 92, def. reg. pol.

4. $\therefore \triangle ABO \cong \triangle CBO$, and $OA = OC$. Why?

Similarly each of the lines OB, OD, ... $= OC$.

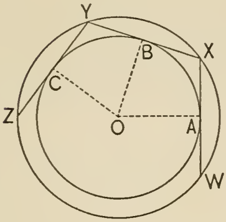
5. $\therefore$ O is the center, and OA, OB, ... are radii. § 108

292. Note. The inscription and circumscription of regular polygons are seen to depend upon the partition of the perigon. Elementary geometry is thus limited to the inscription and circumscription of regular polygons of 2^n, $3 \cdot 2^n$, $5 \cdot 2^n$, $15 \cdot 2^n$ sides; or, since the discovery by Gauss, to polygons the number of whose sides is represented by the product of 2^n and one or more different prime numbers of the form $2^m + 1$.

In addition to regular convex polygons, cross polygons can also be regular, the common five-pointed star being an example.

PROPOSITION XII.

293. Problem. *To inscribe a circle in a given regular polygon.*



Given a regular polygon WXY .....

Required to inscribe a circle in it.

Construction. 1. Circumscribe a circle about it. Prop. XI

2. From center O of this ⊙ draw $OA \perp WX$.
I, prop. XXX

With center O, and radius OA, a ⊙ may be inscribed.

Proof. 1. Draw OB, OC, ... $\perp$ XY, YZ, ...
Then $\because$ OA bisects WX, $\therefore$ A lies between W and X, and so for B, C, ... III, prop. V

2. And $\because$ $WX = XY = \cdots\cdots$, $\therefore$ $OA = OB = \cdots\cdots$
III, prop. VII

3. $\therefore$ if with center O and radius OA a ⊙ is described, then WX, XY, ... will be tangent to the ⊙,
III, prop. IX, cor. 3

and $\therefore$ the ⊙ is inscribed in the polygon.
§ 201, def. inscr. ⊙

Exercises. **440.** Solve prop. XI by bisecting the sides AB, BC by perpendiculars, thus determining O.

441. Inscribe a regular cross pentagon in a circle. (The regular cross pentagon, the pentagram, was the badge of the Pythagorean school.)

Corollaries. 1. *The inscribed and circumscribed circles of a regular polygon are concentric.*

For from step 2 of the construction and step 2 of the proof, O is the center of both circles.

2. *The bisectors of the angles of a regular polygon meet in the common in- and circumcenter.*

For by the proof of prop. XI they meet in O, and by cor. 1 O is the common in- and circumcenter.

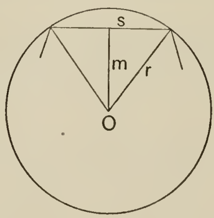
3. *The perpendicular bisectors of the sides of a regular polygon meet in the common in- and circumcenter.* (Why?)

294. Definitions. The radius of the circumscribed circle is called the **radius** of a regular polygon; the radius of the inscribed circle, the **apothem** of that polygon; the common center of the two circles, the **center** of that polygon.

E.g. in the figure below, r is the radius, m the apothem, and O the center of the regular polygon, part of which is shown as inscribed in the circle.

Proposition XIII.

295. Theorem. *The area of a regular polygon equals half the product of the apothem and perimeter.*



Given an inscribed regular polygon, of area a, perimeter p, apothem m.

To prove that $a = \frac{1}{2} mp.$

Proof. Let O be the center and r the radius of the circumscribed circle.

Let t be one of the $\triangle$ formed by joining O to two consecutive vertices, and s a side of the polygon.

Then area t equals $\frac{1}{2}ms$. Why?

$\therefore$ the area of the polygon equals the sum of the areas of the triangles $= \frac{1}{2}m \times$ the sum of the sides $= \frac{1}{2}mp$. Ax. 2

COROLLARIES. 1. *The areas of regular polygons of the same number of sides are proportional to the squares of their apothems, of their radii, or of their sides.*

For $\frac{a}{a'} = \frac{\frac{1}{2}mp}{\frac{1}{2}m'p'} = \frac{mp}{m'p'}$; and from similar $\triangle$ and IV, prop. XX, $\frac{m}{m'} = \frac{r}{r'} = \frac{s}{s'} = \frac{p}{p'}$; $\therefore$ by substitution $\frac{a}{a'} = \frac{m^2}{m'^2} = \frac{r^2}{r'^2} = \frac{s^2}{s'^2}$.

2. *The perimeters of regular polygons of the same number of sides are proportional to their apothems, their radii, or their sides.*

Proved with cor. 1.

Exercises. 442. The distance from the center to a side of the inscribed equilateral triangle equals $r/2$.

443. Draw a diameter AB of a circle with center O; then with center A and radius AO draw an arc cutting the circumference in C, D; draw CD, DB, BC, and prove $\triangle BCD$ equilateral. X

444. The area of an inscribed regular hexagon is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

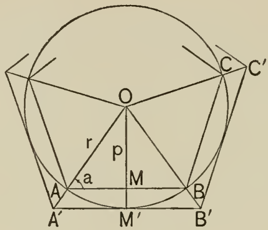
445. Show how, with compasses alone, to divide a circumference into six equal arcs.

446. Prove that if AB, CD, two diameters of a circle, are perpendicular to each other, then $ACBD$ is an inscribed square.

447. Let OX be the perpendicular bisector of line-segment AB at O; lay off on OX, $OD = AO$; and, on DX, lay off $DC = DB$; then prove that C is the center of the $\odot$ circumscribed about the regular octagon of which AB is a side.

4. THE MENSURATION OF THE CIRCLE.

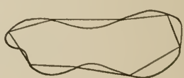
296. Postulate of Limits. *The circle and its circumference are the respective limits which the inscribed and circumscribed regular polygons and their perimeters approach, if the number of their sides increases indefinitely.*



The following may be read by the student in connection with the postulate, although it does not constitute a proof:

1. In the figure, suppose an in- and circumscribed regular n-gon represented. Then each exterior angle equals $\frac{360°}{n}$ in each figure.
2. $\therefore$ each interior angle equals $180° - \frac{360°}{n}$, and $\therefore \angle a = 90° - \frac{180°}{n}$.
3. $\therefore$ if n increases indefinitely, $\angle a \doteq 90°$, and $p \doteq r$.
4. $\therefore$ the inscribed polygon $\doteq$ the circle, and its perimeter $\doteq$ the circumference. Similarly for the circumscribed polygon.

Corollaries. 1. *The circumscribed regular polygon and its perimeter are respectively greater than the circle and its circumference; the inscribed, and its perimeter, less.*



2. *If, on any finite closed curve, n points are assumed equidistant from each other, and each connected with the succeeding point by a straight line, then the curve is the limit which the broken line approaches if n increases indefinitely.*

PROPOSITION XIV.

297. Theorem. *The ratio of the circumference to the diameter of a circle is constant.*

Proof. 1. Suppose any two circles, of circumferences c, c', radii r, r', and diameters d, d', respectively, to have similar regular polygons inscribed in them, of perimeters p, p', respectively.

Then $p : p' = r : r'$, Prop. XIII, cor. 2

$= 2r : 2r' = d : d'$. IV, prop. VIII

2. And $\because$ r, r', d, d' do not change when the number of sides of the polygons is doubled, quadrupled, ..., § 294, def. radius polyg.

and $\because$ $p \doteq c$, and $p' \doteq c'$, § 296, post. of limits

$\therefore c : c' = d : d'$. IV, prop. IX

3. $\therefore c : d = c' : d' =$ the same for any ⊙. IV, prop. III

NOTE. This constant ratio $c : d$ is designated by the symbol π (pi), the initial letter of the Greek word for circumference (*periphereia*). The value of π is discussed in prop. XVII.

COROLLARIES. 1. $c = \pi d$, *or* $2\pi r$.

For if $\frac{c}{d} = \pi$, then $c = \pi d$.

2. *If the radius of a circle is* 1, *then* $c = 2\pi$, *or a semicircumference equals* π.

3. *The circumferences of two circles are proportional to their radii.*

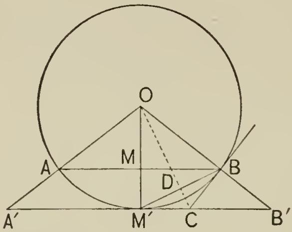
For $\frac{c}{c'} = \frac{2\pi r}{2\pi r'} = \frac{r}{r'}$.

Exercises. 448. Find, in terms of the radius of the circle, r, the side, apothem, and area of the inscribed and circumscribed equilateral triangle.

449. Also of the inscribed and circumscribed square.

Proposition XV.

298. Problem. *Given the sides of the regular inscribed and circumscribed* n-*gons, to find the side of the regular circumscribed* 2 n-*gon.*



Solution. 1. In the figure,

let AB = a side of the regular inscribed n-gon, i_n;
" $A'B'$ = " " " circumscribed " c_n.
Then BM'
= " " " inscribed $2n$-gon, i_{2n};
and BC
= $\frac{1}{2}$ " " " circumscribed " c_{2n}.

2. But ∵ OC bisects $\angle M'OB'$, Why?

$\therefore CB' : M'C = OB' : OM' \ (= OB)$, IV, prop. XIII

$= A'B' : AB$, IV, prop. X, cor. 3

$= c_n : i_n$.

3. $\therefore CB' + M'C : M'C = c_n + i_n : i_n$, IV, prop. V

or $M'B' : M'C = c_n + i_n : i_n$.

4. $\therefore 2\,M'B' : 2\,M'C = c_n + i_n : i_n$. IV, prop. VIII

5. $\therefore c_n : c_{2n} = c_n + i_n : i_n$,

or $$c_{2n} = \frac{c_n \cdot i_n}{c_n + i_n}.$$

PROPOSITION XVI.

299. Problem. *Given the sides of the regular inscribed* n-*gon and the regular circumscribed* 2 n-*gon, to find the side of the regular inscribed* 2 n-*gon.*

Solution. 1. In the figure on p. 218,

$$\triangle M'BM \backsim \triangle CM'D. \qquad \text{Why?}$$

$$\therefore M'B : BM = CM' : M'D. \qquad \text{Why?}$$

2. Or $$i_{2n} : \tfrac{1}{2} i_n = \tfrac{1}{2} c_{2n} : \tfrac{1}{2} i_{2n}. \qquad \text{Why?}$$

3. $$\therefore i_{2n} = \tfrac{1}{2}\sqrt{2\, c_{2n} \cdot i_n}. \qquad \text{Why?}$$

COROLLARIES. 1. *If* p_n, p_{2n}, P_n, P_{2n} *represent the perimeters of the polygons with sides* i_n, i_{2n}, c_n, c_{2n}, *respectively, then* (1) $P_{2n} = \dfrac{2 P_n \cdot p_n}{P_n + p_n}$, *and* (2) $p_{2n} = \sqrt{p_n \cdot P_{2n}}$.

For $c_{2n} = P_{2n}/2n$, $c_n = P_n/n$, $i_{2n} = p_{2n}/2n$, and $i_n = p_n/n$; substitute these in the final steps of props. XV, XVI. From prop. XV,

$$\frac{P_{2n}}{2n} = \frac{P_n/n \cdot p_n/n}{P_n/n + p_n/n} = \frac{P_n \cdot p_n/n}{P_n + p_n}.$$

$$\therefore P_{2n} = \frac{2 P_n \cdot p_n}{P_n + p_n}.$$

From prop. XVI,

$$\frac{p_{2n}}{2n} = \frac{1}{2}\sqrt{2 \cdot \frac{P_{2n}}{2n} \cdot \frac{p_n}{n}}.$$

$$\therefore p_{2n} = \sqrt{p_n \cdot P_{2n}}.$$

2. $c_n = \dfrac{r \cdot i_n}{\sqrt{r^2 - \frac{1}{4} i_n^2}}$, *where* r *is the radius.*

For $c_n : i_n = r : OM = r : \sqrt{r^2 - \left(\dfrac{i_n}{2}\right)^2} = r : \sqrt{r^2 - \frac{1}{4} i_n^2}$.

$$\therefore c_n = \frac{r \cdot i_n}{\sqrt{r^2 - \frac{1}{4} i_n^2}}, \text{ by multiplying by } i_n.$$

Proposition XVII.

300. Theorem. *The approximate value of π is 3.14159 +.*

Proof. 1. In a regular hexagon inscribed in a circle of diameter 1, $i_6 = \frac{1}{2}$, and $\therefore p_6 = 3$. Prop. IX, cor. 1

2. Of the regular hexagon circumscribed about that ⊙, $c_6 = 1 / \sqrt{3}$. Prop. X, cor. 1

3. $\therefore P_6 = 6 \cdot c_6 = 3.4641016 \ldots\ldots$

4. From p_6 and P_6 can be found p_{12} and P_{12}. Props. XV, XVI

5. From p_{12} and P_{12} can be found p_{24} and P_{24}, and so on. Props. XV, XVI

If the process were continued to a 1536-gon, p_{1536} would be found to be 3.1415904, and P_{1536} would be found to be 3.1415970.

6. And $\because c$, or πd, which equals $\pi \cdot 1$ or π, lies between p_n and P_n, however large n may be, § 296, post. of limits, cor. 1

$\therefore \pi$ lies between 3.1415904 and 3.1415970, and is, therefore, approximately 3.14159 +.

Exercises. 450. The diagonals of a regular pentagon cut each other in extreme and mean ratio.

451. If $ABCDE$ is a regular pentagon, and AD cuts BE at P, prove that $AP : AE = AE : AD$.

452. To construct a regular pentagon equal to the sum of two given regular pentagons.

453. Find, in terms of the radius of the circle, r, the side of the inscribed regular pentagon. (Omit unless ex. 433 was taken.)

454. Also of the inscribed and circumscribed regular hexagon.

455. Also of the inscribed and circumscribed regular dodecagon.

456. Also of the inscribed regular decagon. (Depends on ex. 453.)

301. NOTES. The computation in prop. XVII, which the student is not expected to make, is as follows:

No. of sides	p	P
6	3.	3.4641016
12	3.1058285	3.2153903
24	3.1326286	3.1596599
48	3.1393502	3.1460862
96	3.1410319	3.1427146
192	3.1414524	3.1418730
384	3.1415576	3.1416627
768	3.1415838	3.1416101
1536	3.1415904	3.1415970
. . .	. . .	. . .

302. The following historical notes on π are inserted to show the student how the subject of the mensuration of the circle has grown.

The early approximation for π, in use among the ancient people, was 3. See I Kings, vii, 23; II Chron. iv, 2. "What is three hand-breadths around is one hand-breadth through." — *The Talmud.*

Ahmes, however, gave the equivalent of 3.1604.

Archimedes seems to have been the first to employ geometric methods similar to that of props. XV, XVI for approximating π. He announced, "The circumference of a circle exceeds 3 times the diameter by a part which is less than $\frac{1}{7}$, but more than $\frac{10}{71}$, of the diameter."

Hero of Alexandria used both 3 and $3\frac{1}{7}$.

Ptolemy of Alexandria gave $3\frac{17}{120}$.

Aryabhatta found 3.1416, by a method similar to that of prop. XVII.

Brahmagupta used the values of Archimedes; also $\frac{3927}{1250}$ and $\frac{754}{240}$, the last being only another form for Ptolemy's.

Metius gave the easily remembered value 355/113.

Ludolph van Ceulen computed π to the equivalent of over 30 decimal places (the decimal fraction was not yet invented), and wished it engraved on his tomb at Leyden. On this account π is often called in Germany, "the Ludolphian number."

Vega carried it to 140 decimal places.

Dase carried it to 200 decimal places.

Richter carried it to 500 decimal places. More recently Shanks carried it to 707 decimal places.

The symbol π is first used in this sense in Jones's "Synopsis Palmariorum Matheseos," London, 1706.

303. Definition. It is now necessary to extend our idea of **equal surfaces.** The definition at the beginning of Book II, § 142, is true, and it suffices for the cases there under consideration. But when curvilinear figures are compared with rectilinear, it is impossible to cut the surfaces into parts respectively congruent. Hence, we enlarge the definition, thus: *Two surfaces are said to be equal if they have the same numerical measure in terms of a common unit.*

Thus, a circle having an area of $2\,m^2$ would equal a rectangle $2\,m$ long by $1\,m$ broad, even though they could not be cut into parts respectively congruent.

304. Table of Values. The following table of values of expressions involving π will be found useful in computations concerning the circle, sphere, cylinder, cone, etc.:

$\pi = 3.14159$	$\sqrt{\pi} = 1.77245$	$180°/\pi = 57°.29578$
$\pi/4 = 0.78540$	$1/\sqrt{\pi} = 0.56419$	$\pi/180 = 0.01745$
$1/\pi = 0.31831$	$\pi\sqrt{2} = 4.44288$	Approximate values:
$\pi^2 = 9.86960$	$\sqrt{\pi/2} = 1.25331$	$\pi = \frac{22}{7} = 3\frac{1}{7}, \frac{355}{113}$.

The table is repeated, with other tables of value in numerical computations, at the end of this work.

305. Radian Measure of Angles and Arcs. Since if $A =$ any central angle and $a =$ its arc,

$$A : \text{st.}\angle = a : \text{semicircumf.} = a : \pi r.$$
$$\therefore A : \text{st.}\angle/\pi = a : r,$$
$$\text{or } A : 180°/\pi = a : r,$$
$$\text{or } A : 57°.29+ = a : r.$$

That is, the ratio of a central angle to st. $\angle/\pi$ equals the ratio of its arc to an arc of the same length as the radius. Just as the "degree" is the unit for both angle and arc measure, it being understood to be $\frac{1}{360}$ of a perigon in the one case and $\frac{1}{360}$ of a circumference in the other, so a special name is given to st. $\angle/\pi$ and to an arc which equals a radius in length; this name is **radian.** In other words, a radian

is $\frac{1}{\pi}$ of a st. $\angle$, in angle measure, and $\frac{1}{\pi}$ of a semicircumference, or an arc equal to a radius in length, in arc measure.

Since $r = \frac{1}{\pi} 180°$, $\therefore r = 57°.29+$, where r stands for radian.

Also $\because 180° = \pi r$, $\therefore 1° = \frac{\pi r}{180}$ or $\frac{\pi}{180}$ of a radian, or .0174533 of a radian.

In most work in advanced mathematics the radian measure is used exclusively. In common measurements the degree is used. It is necessary in this work to use both.

It is customary to express an angle in radians by the Greek letters α (alpha), β (beta), γ (gamma), ..., the first letters of that alphabet.

306. COROLLARY. *The length of an arc equals the product of the radius by the angle in radians.*

For if a = length of arc, and α = its $\angle$ in radians, then $\frac{a}{c} = \frac{\alpha}{2\pi}$, $\therefore a = \alpha \cdot \frac{2\pi r}{2\pi} = \alpha \cdot r$.

Exercises. 457. Express the following in radians: 10°, 21° 20′, 57°, 58°, 90°.

458. Express the following in degrees: 1.3090 r, .8058 r, .3636 r, .1687 r, .0029 r.

459. Express the following in radians: 100°, 180°, 270°.

460. Express the following in degrees: 3.4907 r, 5.2359 r, 6.2832 r, πr.

461. Find the lengths of arcs of 47° 50′, 61° 20′, 75° 40′, the radius being 10.

462. Given the lengths of the following arcs, to find the radii of the various circles: 75° 10′, 131.19; 32° 20′, 2.822; 4°, .0698.

463. Show that the perimeters of the inscribed and circumscribed squares, the diameter of the circle being 1, are respectively 2.8284271 and 4; hence, find the perimeters of the inscribed and circumscribed regular octagons, and thus show that the value of π may be approximated in this way.

464. The circumferences of certain ⓢ are 43.9823, 84.8230, 128.8053, 185.5340, 204.2035; find the diameters.

Proposition XVIII.

307. Theorem. *The area of a circle equals half the product of its circumference and radius.*

Given a, c, r, the area, circumference, and radius of a circle.

To prove that $a = \frac{1}{2} cr$.

Proof. 1. If a', p represent the area and perimeter of a circumscribed regular polygon, then the apothem of that polygon is r. § 294

2. And $a' = \frac{1}{2} pr$. Prop. XIII

3. But $a' \doteq a$, and $\frac{1}{2} pr \doteq \frac{1}{2} cr$. § 296, post. of limits

4. $\therefore a = \frac{1}{2} cr$. IV, prop. IX, cor. 1

Corollaries. 1. $a = \pi r^2$.

For $c = 2\pi r$.

2. $a = \dfrac{c^2}{4\pi}$. (Why?)

3. *If* s *represents the area of a sector, and* α *its angle in radians, then* $s = r^2\alpha / 2$.

For $s : \pi r^2 = \alpha : 2\pi$. (IV, prop. XVI, cor.)

4. *Of two unequal circles, the greater has the greater circumference.*

For, by cor. 2, $$a = \frac{c^2}{4\pi}.$$

$$\therefore 4\pi a = c^2.$$

$\therefore$ as the area increases, the circumference increases also.

5. *The areas of two circles are proportional to the squares of their radii.*

For $\dfrac{a}{a'} = \dfrac{\pi r^2}{\pi r'^2} = \dfrac{r^2}{r'^2}$

308. HISTORICAL NOTE ON QUADRATURE OF THE CIRCLE. The expression, "to square the circle," means to find the side of a square whose area equals that of a given circle. The solution of this problem by elementary geometry has been proved to be impossible. It nevertheless occupied the attention of many mathematicians before this impossibility was shown, and many ignorant people still attempt it. Some of the Pythagorean school claimed to have solved it, Anaxagoras (died 428 B.C.) wrote upon it, and hundreds of writers since then have discussed the subject. It is closely related to finding a straight line equal to a given circumference ("to rectify the circumference"), and the two depend upon finding the value of π exactly. That π cannot be expressed exactly, nor as the root of a rational algebraic equation, was shown by Lindemann in 1882.

For the mathematical discussion, see Klein's "Famous Problems of Elementary Geometry," translated by the authors. (Boston, Ginn & Co.)

Exercises. **465.** What is the radius of that circle of which the number of square units of area equals the number of linear units of circumference?

466. Also, of which the number of square units of area equals the number of linear units of radius?

467. Give a formula for a in terms of d, and the constant π.

468. A circle equals a triangle of which the base equals the circumference and the altitude equals the radius.

469. Find the areas of circles with radii 5, 7, 21, 35, 47, 50. (In these computations, for uniformity let $\pi = 3.1416$.)

470. Also with diameters 2, 8, 11, 31, 42, 97.

471. Find the radii of circles of areas 78.5398, 2042.8206, 4536.4598.

472. Also the diameters of circles of areas 2123.7166, 3318.3072, 56.745017.

473. Also the circumferences of circles of areas 95.0332, 452.3893.

474. Also the areas of circles of circumferences 267.0354, 191.6372.

475. The area of the ring formed between the circumferences of two concentric circles of radii r_1, r_2, where $r_1 > r_2$, is $\pi(r_1 + r_2)(r_1 - r_2)$.

476. The area of that portion of the ring of ex. 475 cut off by the arms of the central angle α radians is $\frac{1}{2}\alpha(r_1 + r_2)(r_1 - r_2)$; or, if a_1, a_2 are arcs bounding that portion, the area $= \frac{1}{2}(a_1 + a_2)(r_1 - r_2)$.

NOTE. The remainder of the work may be omitted without destroying the integrity of the course.

APPENDIX TO PLANE GEOMETRY.

1. SUPPLEMENTARY THEOREMS IN MENSURATION.

Proposition XIX.

309. Theorem. *If the sides of a triangle are* a, b, c, *and if* $s = \frac{1}{2}(a + b + c)$, $s_1 = s - a$, $s_2 = s - b$, $s_3 = s - c$, *then the area equals* $\sqrt{s \cdot s_1 \cdot s_2 \cdot s_3}$.

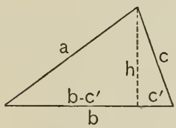


Fig. 1.

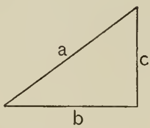


Fig. 2.

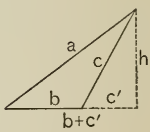


Fig. 3.

Proof. 1. $a^2 = b^2 + c^2 \mp 2\,bc'$, $2\,bc'$ taking the sign $-$ for Fig. 1, $+$ for Fig. 3, and being 0 for Fig. 2. § 159

$\therefore\ c' = \pm\,(b^2 + c^2 - a^2)\,/\,2\,b$, by solving the above equation for c'. Axs. 2, 7

2. But $h^2 = c^2 - c'^2 = (c + c')\,(c - c')$ § 154
$= [c + (b^2 + c^2 - a^2)\,/\,2\,b]\,[c - (b^2 + c^2 - a^2)\,/\,2\,b]$, by substituting the value of c' given in step 1.

$\therefore\ h^2 = (2\,bc + b^2 + c^2 - a^2)\,(2\,bc - b^2 - c^2 + a^2)\,/\,4\,b^2$, by removing parentheses and simplifying.

3. $\therefore\ 4\,b^2h^2 = [(b + c)^2 - a^2]\,[a^2 - (b - c)^2]$, by multiplying by $4\,b^2$ and factoring.

$\therefore\ 4\,b^2h^2 = (b + c + a)(b + c - a)(a + b - c)(a - b + c)$, by factoring still farther.

4. But if $a + b + c = 2s$, as given,

then $b + c - a = 2(s - a) = 2s_1$,

and $a - b + c = 2(s - b) = 2s_2$,

and $a + b - c = 2(s - c) = 2s_3$.

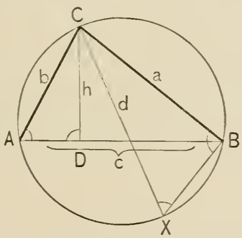
5. $\therefore 4b^2h^2 = 2s \cdot 2s_1 \cdot 2s_2 \cdot 2s_3$. Subst. in 3

6. $\therefore$ area $= \frac{1}{2}bh = \sqrt{s \cdot s_1 \cdot s_2 \cdot s_3}$. V, prop. II, cor. 3

NOTE. This is known as Hero's formula for the area of a triangle. Of course a, b, c represent numerical values as explained under V, prop. II, cor. 2.

PROPOSITION XX.

310. Theorem. *The radius,* r, *of the circle circumscribed about the triangle* abc *of area* t, *equals* abc/4 t.



Proof. 1. Suppose $CX = d$ a diameter, CD (or h) $\perp AB$, and BX drawn.

Then $\triangle ADC \backsim \triangle XBC$, IV, prop. XVII

and $\therefore d : a = b : h$. Why?

2. $\therefore r = ab/2h$. IV, prop. I; ax. 7

3. But $\because \frac{1}{2}hc = t$, V, prop. II, cor. 3

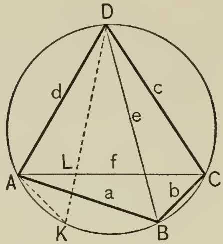
$\therefore r = abc/4t$. Subst. 3 in 2

NOTE. The value of t can be found by Hero's formula.

Exercise. 477. Find the areas of the triangles with sides (1) 13, 14, 15; (2) 3, 5, 8; (3) 7, 10, 18; (4) a, a, a; (5) 3, 4, 5.

Proposition XXI.

311. Theorem. *The product of the diagonals of an inscriptible quadrilateral equals the sum of the products of the opposite sides.*



Given $ABCD$, an inscriptible quadrilateral, with sides a, b, c, d, and diagonals e, f.

To prove that $ef = ac + bd$.

Proof. 1. Let $ABCD$ be inscribed, the sides arranged as in the figure, chord $AK = BC$, and DK drawn cutting AC at L.

Then $\angle ADK = \angle BDC$, and $\angle CAD = \angle CBD$,

and $\therefore \triangle ALD \backsim \triangle BCD$. Why?

2. Also $\because \angle DCL = \angle DBA$, and $\angle LDC = \angle ADB$, III, prop. XI, cor. 1; ax. 2

$\therefore \triangle CDL \backsim \triangle BDA$. Why?

3. From 1, $AL : d = b : e$, or $AL = bd/e$; IV, prop. XX

from 2, $LC : c = a : e$, or $LC = ac/e$. IV, prop. XX

4. $\therefore AL + LC$, or AC, or $f = (ac + bd)/e$. Ax. 2

5. $\therefore ef = ac + bd$. Ax. 6

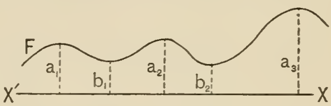
Note. Ptolemy's theorem.

Exercise. 478. Is prop. XXI true when $b =$ zero?

2. MAXIMA AND MINIMA.

312. Definitions. If a geometric magnitude can, by continuous change, increase until a value is reached at which the magnitude begins to decrease, such value is called a **maximum** value; if it can similarly decrease until a value is reached at which it begins to increase, such value is called a **minimum** value.

In general, a magnitude can have more than one maximum or minimum value, as in the annexed figure where a_1, a_2, a_3 represent maximum, and b_1, b_2, minimum values of the ordinates of F. In the elementary geometry of the line and circle, however, only one maximum or minimum exists, so that the words here mean *greatest* and *least*.



E.g. the maximum chord of a circle is the diameter (III, prop. VIII, cor.), and the minimum chord is spoken of as zero, since zero is the limit which constantly decreasing chords of a circle approach.

A magnitude at its maximum value is called *a maximum;* similarly, *a minimum.* *E.g.* a chord of a circle is a maximum when it is a diameter.

313. Figures having equal perimeters are said to be **isoperimetric.**

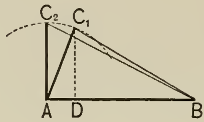
Exercises. 479. Draw a line AB, bisect it at M, and take a point X on AM; then show that $AX^2 + XB^2 = 2\,AM^2 + 2\,XM^2$, and that this is a minimum when $XM = 0$; hence show that the sum of the squares on the two segments of a given line is a minimum when the segments are equal.

480. Also that $AX \cdot XB = MB^2 - XM^2$, and that this is a maximum when $XM = 0$; that is, that the rectangle of the two segments into which a given line can be divided is a maximum when the given line is bisected.

481. If the diagonals of an inscribed quadrilateral are perpendicular to each other, then the sum of the products of the two opposite sides equals twice the area of the quadrilateral.

Proposition XXII.

314. Theorem. *Of all triangles formed with the same two given sides, that is the maximum whose sides contain a right angle.*



Given the $\triangle s$ ABC_1, ABC_2, with $AC_1 = AC_2$, and $AC_2 \perp AB$.

To prove that $\triangle ABC_2 > \triangle ABC_1$.

Proof. Suppose $C_1D \perp AB$.

Then $AC_1 > DC_1$, I, prop. XX

and $\therefore$ its equal $AC_2 > DC_1$.

$\therefore \triangle ABC_2 > \triangle ABC_1$, II, prop. II, cor. 3

since they have the same bases but different altitudes.

Exercises. 482. Find in radians the angle α of a sector of a circle of radius r, such that the number of square units of its area equals the number of linear units of its entire perimeter.

483. Interpret the result of ex. 482 for $r = 2\left(1 + \frac{1}{\pi}\right)$. Discuss it for $r \nless 2$. Discuss it for $r < 2\left(1 + \frac{1}{\pi}\right)$.

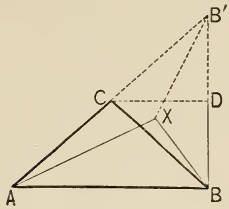
484. In the Sulvasutras, early semi-theological writings of the Hindus, it is said: "Divide the diameter into 15 parts and take away 2; the remainder is approximately the side of the square equal to the circle." From this compute their value of π.

485. On AB describe a semicircle, and in it inscribe the isosceles triangle ABC; on BC and CA describe semicircles opposite the $\triangle ABC$. Show that $\triangle ABC =$ the sum of the two lunes thus formed. (The lunes of Hippocrates.)

486. Six lights are placed regularly on the circumference of a circle of radius 21 ft.; what are the distances of each from each of the others? (To 0.01.)

PROPOSITION XXIII.

315. Theorem. *Of all isoperimetric triangles on the same base the isosceles is the maximum.*



Given two isoperimetric $\triangle ABC$ and ABX, $\triangle ABC$ being isosceles, with $AC = BC$.

To prove that $\triangle ABC > \triangle ABX$.

Proof. 1. On AC produced, let $CB' = AC$; draw $B'B$, $B'X$; suppose $CD \parallel AB$.

Then $\because AC = CB'$,

$\therefore BD = DB'$. I, prop. XXVII, cor. 2

2. And $\because CB = AC$,

$\therefore CB = CB'$, and $CD \perp BB'$. Why? Ax. 1

3. $\therefore AC + CB = AC + CB' < AX + XB'$.
Ax. 2; I, prop. VIII

4. $\because AX + XB = AC + CB$, Why?

$\therefore AX + XB < AX + XB'$,

and $\therefore XB < XB'$. Why?

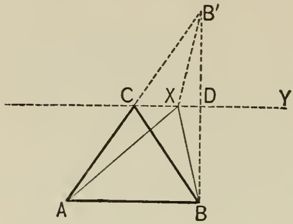
5. $\therefore X$ and AB lie on the same side of CD,
I, prop. XX, cor. 3

and $\therefore \triangle ABC > \triangle ABX$. II, prop. II, cor. 3

COROLLARY. *Of all isoperimetric triangles, that which is equilateral is the maximum.* (Why?)

Proposition XXIV.

316. Theorem. *Of all triangles having the same base and area, the isosceles has the minimum perimeter.*



Given the $\triangle ABC$ and ABX having the same base and area, with $AC = BC$.

To prove that perimeter $ABC <$ perimeter ABX.

Proof. 1. Suppose $CY \parallel AB$; AC produced so that $CB' = AC$; $B'X$ drawn; and $B'B$ drawn cutting CY at D.

Then $\because \triangle ABC = \triangle ABX$,

$\therefore CY$ passes through X. II, prop. II, cor. 4

2. And $\because AC = CB'$,

$\therefore BD = DB'$. I, prop. XXVII, cor. 2

3. And $\because \triangle BDC \cong \triangle B'DC$, I, prop. XII

$\therefore CD \perp BB'$, Why?

and $\therefore XB = XB'$. I, prop. XX

4. But $AC + CB' < AX + XB'$, I, prop. VIII

and $\therefore AC + CB < AX + XB$.

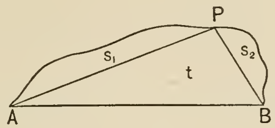
5. $\therefore$ perim. $ABC <$ perim. ABX. Why?

Corollary. *Of all equal triangles, that which is equilateral has the minimum perimeter.*

For whatever side is taken as the base, the perimeter is less if the other two sides are equal.

PROPOSITION XXV.

317. Theorem. *If the ends of a line of given length are joined by a straight line, and the area of the figure enclosed is a maximum, it takes the form of a semicircle.*



Given a line APB (the curve in the figure), of given length, and AB joining its end-points.

To prove that, if the area of the figure ABP is a maximum, ABP is a semicircle.

Proof. 1. Let P be any point on the line; then joining A and P, B and P, let the segments cut off by AP, BP be called s_1, s_2, and $\triangle ABP$ called t, as in the figure.

Then $\angle P$ is a right angle; for if not, without changing s_1, s_2, the area of t could be increased by making $\angle P$ right. Prop. XXII

2. But this is impossible if ABP is a maximum, and similarly for any other point on APB. Why?

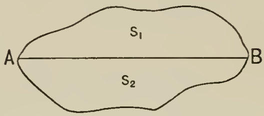
3. $\therefore$ the area enclosed is a maximum when the line connecting A and B subtends a right angle at every point on the curve.

NOTE. It will be seen that examples of maxima or minima involve also the idea of *symmetry* (§ 68). This fact is of value in solving problems in maxima and minima.

Exercise. 487. Given the points A, B, on the same side of line $X'X$, to find on $X'X$ a point P such that $\angle X'PA = \angle BPX$. Prove that $AP + PB$ is the shortest path from A to $X'X$ and back to B. (Reflected ray of light.)

Proposition XXVI.

318. Theorem. *Of all isoperimetric plane figures the maximum is a circle.*

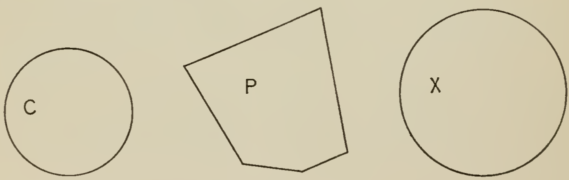


Proof. Suppose A, B points bisecting the given perimeter, AB cutting the figure into two segments, s_1, s_2.

Then s_1, s_2 are maxima when they are semicircles, and AB is a diameter. Why?

Proposition XXVII.

319. Theorem. *Of all equal plane figures the circle has the minimum perimeter.*



Given circle $C =$ plane figure P.

To prove that circumference $C <$ perimeter P.

Proof. 1. Suppose X a circle of circumference equal to perimeter P.

Then $P < X$, Prop. XXVI

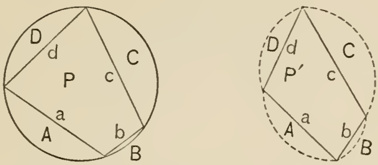
and $\therefore C < X$. Subst.

2. $\therefore$ circumference $C <$ circumference X, § 307, cor. 4

and $\therefore$ circumference $C <$ perimeter P. Subst.

PROPOSITION XXVIII.

320. Theorem. *A polygon with given sides is a maximum when it is inscriptible.*



Given two polygons, P and P', with given sides a, b, c ....., P being inscribed in a circle, and P' not inscriptible.

To prove that $P > P'$.

Proof. 1. Name the circular segments on a, b, ... (opposite P), A, B, ... ; suppose congruent segments constructed on a, b, ... (opposite P').

Then $P + A + B + \cdots > P' + A + B + \cdots$ Prop. XXVI

2. $\therefore P > P'$. Why?

Exercises. 488. If the diagonals of a parallelogram are given, its area is a maximum when it is a rhombus.

489. What is the minimum line from a given point to a given line? Where has this been proved?

490. Into what two parts must a given number be divided so that the product of those parts shall be a maximum? (Compare ex. 479.)

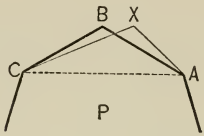
491. As a corollary to ex. 479, show that of isoperimetric rectangles the square is the maximum.

492. Find the point in a given straight line such that the tangents drawn from it to a given circle contain the maximum angle.

493. A straight ruler, 1 foot long, slips between the two edges of the floor (the edges making a right angle). Find the position of the ruler when the triangle formed by the edges and ruler is a maximum; also the area of that triangle.

Proposition XXIX.

321. Theorem. *Of all isoperimetric polygons of a given number of sides, the maximum is regular.*



Given P, the maximum polygon of a given perimeter and a given number of sides.

To prove that P is regular.

Proof. 1. Any two adjacent sides, AB, BC, must be equal. For if unequal, as AX, XC, then $\triangle AXC$ could be replaced by $\triangle ABC$, having $AB = BC$, thus enlarging P without changing the perimeter. But this is impossible because P is a maximum. Prop. XXIII

2. And hence P is inscriptible because its sides are given. Prop. XXVIII

3. $\therefore P$ is regular. V, prop. IX, cor. 2

Exercises. 494. Considering only the relation of space enclosed to amount of wall, what would be the most economical form for the ground plan of a house?

495. Of all triangles in a given circle, what is the shape of the one having the greatest area? Prove it.

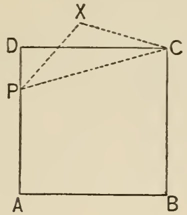
496. Through a point of intersection of two circumferences draw the maximum line terminated by the two circumferences.

497. Of all triangles of a given base and area, the isosceles has the greatest vertical angle.

498. Draw the minimum straight line between two non-intersecting circumferences.

PROPOSITION XXX.

322. Theorem. *Of two isoperimetric regular polygons, that having the greater number of sides is the greater.*



Proof. 1. Let $ABCD$ be a square, P a point on DA, $\triangle PCX$ isoperimetric with $\triangle PCD$ and having $CX = PX$.

Then $\triangle PCX > \triangle PCD$, Prop. XXIII

and $\therefore$ pentagon $ABCXP > \square ABCD$. Ax. 4

2. But pentagon $ABCXP$ would, with the same perimeter, be greater if it were regular. Prop. XXIX

3. $\therefore$ a regular pentagon is greater than an isoperimetric square. Similarly, a regular hexagon would be greater than an isoperimetric regular pentagon, and so on.

Exercises. 499. A cross-section of a bee's cell is a regular hexagon. Show that this is the best form for securing the greatest capacity with a given amount of wax (perimeter).

500. Find the maximum rectangle inscribed in a given semicircle.

501. Find the minimum square inscribed in a given square.

502. Draw the minimum tangent from a variable point in a given line to a given circle.

503. What is the area of the largest triangle that can be inscribed in a circle of radius 5?

504. Given a square of area 1. Find the area of an isoperimetric (1) equilateral triangle, (2) regular hexagon, (3) circle.

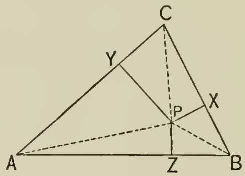
3. CONCURRENCE AND COLLINEARITY.

Proposition XXXI.

323. Theorem. *If* X, Y, Z *are three points on the sides* a, b, c, *respectively, of a triangle* ABC, *such that the perpendiculars to the sides at these points are concurrent, then*

$$(BX^2 - XC^2) + (CY^2 - YA^2) + (AZ^2 - ZB^2) = 0;$$

and conversely.



Proof. Let P be the point of concurrence, and draw PA, PB, PC.

Then $(BX^2 - XC^2) + (CY^2 - YA^2) + (AZ^2 - ZB^2)$ $= PB^2 - PC^2 + PC^2 - PA^2 + PA^2 - PB^2 = 0$, for $BX^2 - XC^2 = (BP^2 - PX^2) - (PC^2 - PX^2) = BP^2 - PC^2$, and so for the rest.

Conversely: 1. Suppose the $\perp$s from X, Y, to meet at P; and suppose $PZ' \perp c$.

Then as above,
$(BX^2 - XC^2) + (CY^2 - YA^2) + (AZ'^2 - Z'B^2) = 0.$

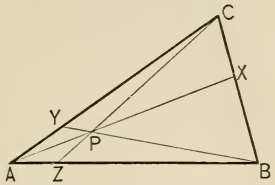
2. But $(BX^2 - XC^2) + (CY^2 - YA^2) + (AZ^2 - ZB^2) = 0$, and $AZ'^2 - Z'B^2 = AZ^2 - ZB^2$. Why?

3. $\therefore AZ'^2 - AZ^2 = Z'B^2 - ZB^2$; but these differences have opposite signs and cannot be equal unless each is zero.

4. $\therefore Z$ must coincide with Z'.

PROPOSITION XXXII.

324. Theorem. *If three lines,* x, y, z, *drawn from the vertices of triangle* ABC *to meet* a, b, c *in* X, Y, Z, *are concurrent, then* $\frac{AZ}{ZB}\cdot\frac{BX}{XC}\cdot\frac{CY}{YA}=1$; *and conversely.*



Proof. 1. Let P be the point of concurrence. Then $\because$ $\triangle$ APC, PBC have the base PC, they are proportional to their altitudes, and $\therefore$ to AZ, ZB. Why ?

2. $$\therefore \frac{AZ}{ZB}=\frac{\triangle APC}{\triangle PBC},$$

and $$\frac{BX}{XC}=\frac{\triangle BPA}{\triangle APC},$$

and $$\frac{CY}{YA}=\frac{\triangle PBC}{\triangle BPA}.$$

3. $$\therefore \frac{AZ}{ZB}\cdot\frac{BX}{XC}\cdot\frac{CY}{YA}=1.$$ Ax. 6

CONVERSELY: Let CP meet c in Z'; then as above,

4. $$\frac{AZ'}{Z'B}\cdot\frac{BX}{XC}\cdot\frac{CY}{YA}=1.$$

5. But $$\frac{AZ}{ZB}\cdot\frac{BX}{XC}\cdot\frac{CY}{YA}=1.$$ Given

6. $$\therefore \frac{AZ'}{Z'B}=\frac{AZ}{ZB}.$$

7. $\therefore$ Z' must coincide with Z. IV, prop. XI, cor.

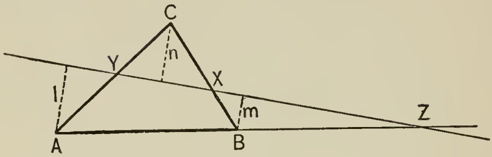
NOTE. Ceva's theorem.

Proposition XXXIII.

325. Theorem. *If three points,* X, Y, Z, *lying respectively on the three sides* a, b, c *of triangle* ABC, *are collinear, then*

$$\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = -1;$$

and conversely.



Proof. 1. Let l, m, n be perpendiculars from A, B, C on XY. Then by similar $\triangle$s, ZB being here negative,

$$\frac{AZ}{ZB} = \frac{l}{-m}.$$

2. And similarly, $\frac{BX}{XC} = \frac{m}{n}$,

and $\frac{CY}{YA} = \frac{n}{l}$.

3. $\therefore \frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = -1.$ Ax. 6

Conversely: Let XY meet AB in Z'; then as above,

4. $\frac{AZ'}{Z'B} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = -1.$

5. But $\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = -1.$ Given

6. $\therefore \frac{AZ'}{Z'B} = \frac{AZ}{ZB}.$

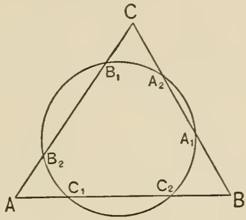
7. $\therefore Z'$ must coincide with Z. IV, prop. XI, cor.

Note. Menelaus's theorem.

PROPOSITION XXXIV.

326. Theorem. *If a circumference intersects the sides,* a, b, c, *of a triangle* ABC, *in the points* A_1 *and* A_2, B_1 *and* B_2, C_1 *and* C_2, *respectively, then*

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} \cdot \frac{AC_2}{C_2B} \cdot \frac{BA_2}{A_2C} \cdot \frac{CB_2}{B_2A} = 1.$$



Proof. 1. $AC_1 \cdot AC_2 = B_1A \cdot B_2A$, Why?

and $BA_1 \cdot BA_2 = C_1B \cdot C_2B$,

and $CB_1 \cdot CB_2 = A_1C \cdot A_2C$.

2. ∴ by axs. 6 and 7, the above result follows.

NOTE. This theorem, known as Carnot's theorem, is not a proposition in concurrence or collinearity. It is introduced as leading to the proof of the very celebrated theorem following, one commonly known as the Mystic Hexagram, discovered by Pascal at the age of 16.

The theorem is also easily proved when the triangle is inscribed or circumscribed.

Exercises. 505. By means of Ceva's theorem, prove that the three medians of a triangle are concurrent.

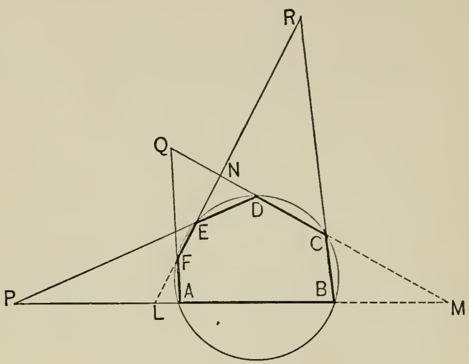
506. Also, that the bisectors of the three interior angles of a triangle are concurrent.

507. Also, that the bisectors of two exterior and of the other interior angles of a triangle are concurrent.

508. Also, that the perpendiculars from the vertices of a triangle to the opposite side are concurrent.

Proposition XXXV.

327. Theorem. *If the opposite sides of an inscribed hexagon intersect, they determine three collinear points.*



Given an inscribed hexagon, $ABCDEF$, such that BA and DE meet at P, CD and AF at Q, BC and FE at R.

To prove that P, Q, R are collinear.

Proof. 1. Call the $\triangle$ determined by AB, CD, and EF, LMN, as in the figure.

Then from Menelaus's theorem,

$$\frac{LP}{PM}\cdot\frac{MD}{DN}\cdot\frac{NE}{EL}=-1,$$

and
$$\frac{MQ}{QN}\cdot\frac{NF}{FL}\cdot\frac{LA}{AM}=-1,$$

and
$$\frac{NR}{RL}\cdot\frac{LB}{BM}\cdot\frac{MC}{CN}=-1.$$

2. $\therefore$ By multiplying and recalling Carnot's theorem,

$$\frac{LP}{PM}\cdot\frac{MQ}{QN}\cdot\frac{NR}{RL}=-1.$$

3. $\therefore$ by Menelaus's theorem, P, Q, R are collinear.

MISCELLANEOUS EXERCISES.

509. Show that the following is a special case of prop. XXXI: The perpendicular bisectors of the sides of a triangle are concurrent.

510. Also, the perpendiculars from the vertices of a triangle to the opposite sides are concurrent.

511. If three circumferences intersect in pairs, the common chords are concurrent.

512. By means of Menelaus's theorem, prove that the points in which the three bisectors of the exterior angles of a triangle meet the opposite sides are collinear.

513. Also, that the points in which the two bisectors of two interior angles of a triangle and the other exterior angle meet the opposite sides are collinear.

514. The orthocenter, O, of $\triangle ABC$ is determined by the perpendiculars AD, BE. Prove that $AO \cdot OD = BO \cdot OE$.

515. Draw a circle with a central right angle AOB, A and B being on the circumference; bisect $\angle AOB$ by OM, meeting $\widehat{AB}$ at M; draw $MP \perp OA$; then see if the following is true in general: $\widehat{AB} =$ chord $AB + PA$. (Consider special cases, $\widehat{AB} = 120°$, $180°$, $360°$.)

516. Given the base and the vertical angle of a triangle; construct it so that its area shall be a maximum.

517. AB is a diameter of a circle of center O; from any point P on the circumference, PC is drawn perpendicular to AB; from C a perpendicular CE is drawn to OP. Prove that PC is a mean proportional between OA and PE.

518. On side a of $\triangle ABC$, point P is taken such that $\angle PAC = \angle B$. Prove that $CP : CB = AP^2 : AB^2$. Investigate for three cases, $\angle A <, =, > \angle B$.

519. ABC is a triangle right-angled at C; $CD \perp c$. Prove that $AD : DB = CA^2 : BC^2$.

520. If O, O' are the centers of two fixed circles, such that the circumference of O' passes through O, and if a tangent to circumference of O at T cuts circumference of O' at X, Y, then $OX \cdot OY$ is constant. (If the center-line meets the circumference of O' at A, $\triangle XTO \sim \triangle AYO$.)

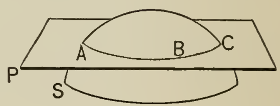
521. If O is the orthocenter of triangle ABC, and A', B', C' are the mid-points of a, b, c; M_a, M_b, M_c are the mid-points of AO, BO, CO; P_a, P_b, P_c are the feet of the perpendiculars from A, B, C to a, b, c; prove that A', B', C', M_a, M_b, M_c, P_a, P_b, P_c are concyclic. (The "Nine Points Circle.")

SOLID GEOMETRY.

BOOK VI. — LINES AND PLANES IN SPACE.

1. THE POSITION OF A PLANE IN SPACE. THE STRAIGHT LINE AS THE INTERSECTION OF TWO PLANES.

328. Definitions. Through three points, not in a straight line, any number of surfaces may be imagined to pass.



For example, through the points A, B, C the surfaces P and S may be imagined to pass.

329. A plane surface (also called a *plane*) is a surface which is determined by any three of its points not in a straight line.

In the figure, P represents a plane, for it is determined by the points A, B, C. But S does not represent such a surface.

A plane is, of course, supposed to be indefinite in extent.

This definition, and the following postulates, are repeated, for convenience, from the Plane Geometry.

In drawing a figure it should be remembered that a plane, like a line, has no thickness, and that it is indefinite in extent. Nevertheless, it aids the eye in understanding the figure, if we represent the plane as a rectangle, lying in perspective, and having a slight thickness.

Exercises. 522. Show that if there are given four points in space, no three being collinear, the number of distinct straight lines determined by them is six; if there are five points, the number of lines is ten.

523. Hold two pencils in such a way as to show that a plane cannot, in general, contain two straight lines taken at random in space.

330. Postulates of the Plane. (See § 29.)

1. *Three points not in a straight line determine a plane.*

2. *A straight line through two points in a plane lies wholly in the plane.*

3. *A plane may be passed through a straight line and revolved about it so as to include any assigned point in space.*

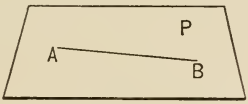
4. *A portion of a plane may be produced.*

5. *A plane is divided into two parts by any one of its straight lines, and space is divided into two parts by any plane.*

331. Solid Geometry treats of figures whose parts are not all in one plane.

PROPOSITION I.

332. Theorem. *A plane is determined by a straight line and a point not in that line.*



Given the line AB, and the point P not in that line.

To prove that AB and P determine a plane.

Proof. 1. Only one plane contains pts. A, B, and P. § 330, 1

(§ 330, 1. Three points not in a straight line determine a plane.)

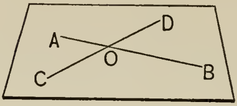
2. And that plane contains line AB. § 330, 2

(§ 330, 2. A straight line joining two points in a plane lies wholly in the plane.)

3. ∴ only one plane contains AB and P.

333. Definition. Lines or points which lie in the same plane are said to be **coplanar**.

COROLLARIES. 1. *A plane is determined by two intersecting lines.*



Let the lines AB, CD intersect at O.

Then only one plane contains AB and C. Prop. I

And that plane contains the point O, for O lies in the line AB. § 330, 2

And since that plane contains C and O, it contains CD. § 330, 2

2. *A plane is determined by two parallel lines.*

For the parallels lie in one plane, by definition (§ 82).

And only one plane can contain these parallels, since a plane is determined by either line and any point of the other.

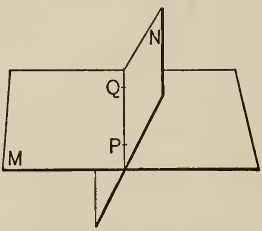
Draw the figure.

3. *If a plane contains one of two parallel lines and any point of the other, it contains both parallel lines.*

For it must be identical with the plane determined by the two parallels; otherwise more than one plane could contain either parallel and any point in the other.

PROPOSITION II.

334. Theorem. *The intersection of two planes is a straight line.*



Given two intersecting planes, M, N.

To prove that their intersection is a straight line.

Proof. 1. Let P be a point common to M and N.

Then a pencil of lines through P, in the plane N, must lie partly on one side of M and partly on the other, because M divides space into two parts. § 330, 5

2. Hence, in general, a line connecting a point in the pencil on one side of M, with a point on the other side, must cut M at some other point than P, — say at Q.

3. Then M and N have two points in common.

4. Then every point in the straight line through P and Q lies in plane M, § 330, 2
and also in plane N, for the same reason.

5. $\therefore$ the straight line PQ is common to both planes.

6. If there were any point not in PQ, common to M and N, the planes would coincide. Prop. I

COROLLARY. *A point common to two planes lies in their line of intersection.*

Proved in step 6.

Exercises. 524. State the four methods, already mentioned, of determining a plane.

525. Is it possible for three planes to have a straight line in common? Draw a figure to illustrate.

526. If two planes have three points in common, will they necessarily coincide?

527. Four planes, no three containing the same line, intersect in pairs; how many straight lines do they determine by their intersections?

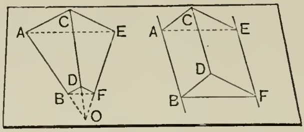
528. What is the only rectilinear polygon that is necessarily plane? Why?

529. Prove that all transversals of two parallel lines are coplanar with the parallels.

530. What is the reason that a three-legged chair is always stable on the floor while a four-legged one may not be?

Proposition III.

335. Theorem. *If three planes, not containing the same line, intersect in pairs, the three lines of intersection are either concurrent or parallel.*



Given planes AD, CF, EB, intersecting in AB, CD, EF.

To prove that AB, CD, EF are either concurrent or parallel.

Proof. Case I. If CD meets AB, as at O, to show the three lines concurrent.

1. $\because O$ is in AB, it is in plane EB. § 330, 2
2. Similarly, $\because O$ is in CD, it is in plane CF. Why ?
3. $\because O$ is in planes EB and CF, EF passes through O. Prop. II, cor.
4. $\therefore AB$, CD, EF are concurrent in O.

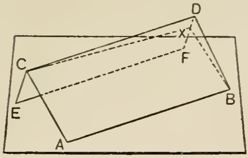
Case II. If $CD \parallel AB$, to show the three lines parallel.

1. If AB were not $\parallel EF$, CD would pass through their common point. Case I
2. But this is impossible, for $CD \parallel AB$. Given
3. If CD were not $\parallel EF$, AB would pass through their common point. Case I
4. But this is impossible, for $CD \parallel AB$. Given
5. $\therefore$ as no two can meet, and as each pair is coplanar, they are parallel. Def. $\parallel$ lines

Corollary. *If two intersecting planes pass through two parallel lines, their intersection is parallel to these lines.*

PROPOSITION IV.

336. Theorem. *Lines parallel to the same line are parallel to each other.*



Given $AB \parallel EF$, $CD \parallel EF$.

To prove that $AB \parallel CD$.

Proof. 1. AB and EF determine a plane. Prop. I, cor. 2

(A plane is determined by two $\parallel$ lines.)

2. CD and EF determine a plane. Why?

3. AB and any point C of CD determine a plane. Prop. I

4. Suppose this last plane to intersect plane ED in CX, another line than CD.

Then CX would be $\parallel$ to both EF and AB. Prop. III, cor.

(If two intersecting planes pass through two $\parallel$ lines, their intersection is $\parallel$ to these lines.)

5. But $\because CD \parallel EF$, this is impossible. Post. of parallels

(§ 85. Two intersecting straight lines cannot both be $\parallel$ to the same straight line.)

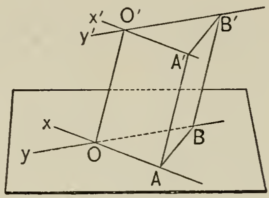
6. $\therefore CD$ is the intersection of the planes through AB and C, and EF and C,

and $\therefore CD \parallel AB$. Prop. III, cor.

Exercise. 531. Why will not the proof of this theorem as given in plane geometry apply to this case in solid geometry?

Proposition V.

337. Theorem. *If two intersecting lines are respectively parallel to two others, the angles made by the first pair are equal or supplemental to those made by the second pair.*



Given two intersecting lines x, y, respectively parallel to two other lines x', y'.

To prove that the angles made by x and y are equal or supplemental to those made by x' and y'.

Proof. 1. Suppose the intersections O and O' are joined, and from any points A, B, on x, y, parallels to OO' are drawn.

2. $\because OO'$, x, and x' are coplanar (Why ?), the parallel from A meets x' as at A'. Similarly, B' is fixed. Prop. I, cor. 3

3. Draw AB, $A'B'$.

$\because AA' \parallel OO'$, and $BB' \parallel OO'$, $\therefore AA' \parallel BB'$. Prop. IV

4. $\because OA'$, OB' are ▱s, $\therefore AA' = OO' = BB'$. I, prop. XXIV

5. $\therefore ABB'A'$ is a ▱. I, prop. XXV

6. $\therefore OA = O'A'$, $OB = O'B'$, $AB = A'B'$. I, prop. XXIV

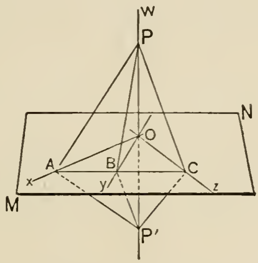
7. $\therefore \triangle ABO \cong \triangle A'B'O'$, and $\angle AOB = \angle A'O'B'$. I, prop. XII

After proving one pair of angles equal, the rest are evidently equal or supplemental by the theorems concerning vertical and supplemental angles.

2. THE RELATIVE POSITION OF A LINE AND A PLANE.

PROPOSITION VI.

338. Theorem. *If a line is perpendicular to each of two intersecting lines, it is perpendicular to every other line lying in their plane and passing through their point of intersection.*



Given x and z, two lines intersecting at O, and w perpendicular to x and to z; also y, any line through O coplanar with x, z.

To prove that $w \perp y$.

Proof. 1. On w suppose $OP' = PO$; let any transversal cut x, y, z at A, B, C; join P and P' with A, B, C.

Then $AP = AP'$, and $CP = CP'$. I, prop. XX, cor. 5

2. And $\because AC \equiv AC$,

$\therefore \triangle ACP \cong \triangle ACP'$. I, prop. XII

3. $\therefore$ by folding $\triangle ACP$ over AC as an axis, it can be brought to coincide with $\triangle ACP'$. § 57

4. $\therefore \triangle BOP \cong \triangle BOP'$, I, prop. XII

and $\angle POB$ is a rt. $\angle$, and $w \perp y$. Why?

339. Definitions. A line is said to be **perpendicular to a plane** when it is perpendicular to every line in that plane which passes through its *foot*, — *i.e.* the point where it meets the plane. The plane is then said to be **perpendicular to the line.**

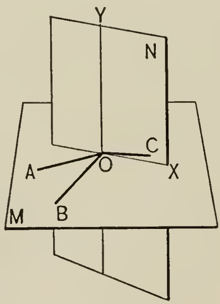
If a line meets a plane, and is not perpendicular to it, it is said to be **oblique to the plane.**

Corollaries. 1. *If a line is perpendicular to each of two intersecting lines, it is perpendicular to their plane.*

2. *The locus of points equidistant from two given points is the plane bisecting at right angles the line joining those points.*

Proposition VII.

340. Theorem. *If a line is perpendicular to each of three concurrent lines at their point of concurrence, the three lines are coplanar.*



Given $OY \perp OA, OB, OC.$

To prove that OA, OB, OC are coplanar.

Proof. 1. Suppose M the plane determined by OA, OB; and N the plane determined by OY, OC.

Suppose that OC is not in M, and call OX the intersection of M and N.

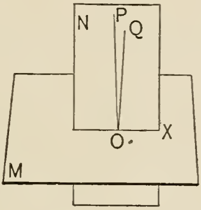
2. Then must $OY \perp OX$. Prop. VI

3. But $\because OY \perp OC$, this is impossible. Prel. prop. II

4. $\therefore$ it is absurd to suppose OC not in M with OA and OB.

COROLLARIES. 1. *Lines perpendicular to the same line at the same point are coplanar.*

2. *Through a given point in a plane there cannot be drawn more than one line perpendicular to that plane.*



Suppose OP and $OQ \perp$ plane M. Then each would be perpendicular to OX, the line of intersection of their plane N with the given plane M, thus violating prel. prop. II.

3. *Through a given point in a line there cannot be drawn more than one plane perpendicular to that line.*

For if two planes could be drawn perpendicular to the line, then three lines in each would be perpendicular to the given line, and hence the two planes would coincide.

Exercises. 532. Prove that if the hand of a clock is perpendicular to its moving axle, it describes a plane in its revolution. Prove the converse.

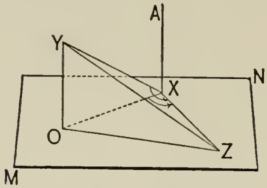
533. How many straight lines are determined by six points, three being collinear?

534. How many planes in general are determined by four points in space, no three being collinear?

535. In the left-hand figure of prop. III, suppose point O to move farther from BDF, and to continue to do so indefinitely. What is the limiting figure which the left-hand figure is approaching?

PROPOSITION VIII.

341. Theorem. *Lines perpendicular to the same plane are parallel.*



Given OY, $XA \perp$ plane MN at O, X.

To prove that $OY \parallel XA$.

Proof. It is necessary first to show that OY, XA are coplanar; then that they are $\perp$ to OX.

1. Let $XZ \perp OX$ in plane MN, and $= OY$.

   Draw OZ, ZY, XY.

2. Then $\because XZ = OY$, $OX \equiv OX$,

   and $\angle XOY = \angle OXZ =$ rt. $\angle$,

3. $\therefore \triangle XOY \cong \triangle OXZ$,

   and $OZ = XY$. I, prop. I

4. And $\because ZY \equiv ZY$, $\therefore \triangle XYZ \cong \triangle OZY$,

   and $\angle YXZ = \angle ZOY =$ rt. $\angle$. I, prop. XII

   $\therefore XA$, XY, XO are coplanar. Why?

5. $\therefore YO$ lies in that same plane. § 330, 2

6. But $\because YO$ and $AX \perp OX$, § 339

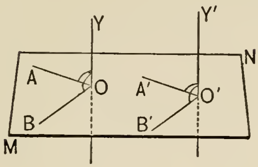
   $\therefore YO \parallel AX$, and similarly for all other $\perp$s.

   I, prop. XVI, cor. 3

COROLLARY. *From a point outside of a plane, not more than one line can be drawn perpendicular to that plane.*

PROPOSITION IX.

342. Theorem. *If one of two parallel lines is perpendicular to a plane, the other is also.*



Given $OY \parallel O'Y'$, $OY \perp$ plane MN at O, and $O'Y'$ meeting plane MN at O'.

To prove that $O'Y' \perp MN$.

Proof. 1. Let OA, OB be any lines from O, in MN, $O'A' \parallel OA$, and $O'B' \parallel OB$.

2. Then $\angle$s YOA, YOB are rt. $\angle$s. § 339

3. But $\angle YOA = \angle Y'O'A'$, $\angle YOB = \angle Y'O'B'$. Prop. V

4. $\therefore$ $\angle$s $Y'O'A'$, $Y'O'B'$ are also rt. $\angle$s, Prel. prop. I
and $O'Y' \perp MN$. § 339

343. Definitions. The **projection of a point** on a plane is the foot of the perpendicular through that point to the plane.

The **projection of a line** on a plane is the locus of the projections of all of its points.

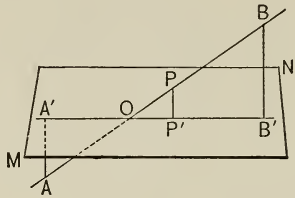
Exercises. 536. Are lines which make equal angles with a given line always parallel? (Answer by drawing figures to illustrate.)

537. Show how to determine the perpendicular to a plane, through a given point, by the use of two carpenter's squares.

538. Prove prop. VI on the following outline: Assume B on y, and draw ABC so that $AB = BC$ (How is this done?); prove $2 \cdot PB^2 + 2 \cdot BC^2 = PA^2 + PC^2 = 2 \cdot PO^2 + OC^2 + OA^2 = 2 \cdot PO^2 + 2 \cdot OB^2 + 2 \cdot BC^2$; $\therefore PB^2 = PO^2 + OB^2$; $\therefore \angle POB$ is a rt. $\angle$.

Proposition X.

344. Theorem. *The projection of a straight line on a plane is the straight line which passes through the projections of any two of its points.*



Given A', P', B', the projections of A, P, B, points in the line AB, on the plane MN.

To prove that the projection of AB is the straight line $A'B'$.

Proof. 1. $AA' \parallel BB' \parallel PP'$. Why?

2. $\therefore A, A', B, B'$ are coplanar. Prop. I, cor. 2

3. $\therefore P$ is in that same plane. Why?

4. $\therefore PP'$ is in that same plane. Prop. I, cor. 3

5. $\therefore A'$, P', B' are collinear. Prop. II

6. Also any other point in $A'B'$ is the projection of some point in AB. For a $\perp$ to MN from such a point is $\parallel$ to AA' (§ 341) and lies in plane $AA'B'B$ (§ 82), and therefore meets AB in some point.

345. Definitions. The smallest angle formed by a line and its projection on a plane is called the **inclination** of the line to the plane or **the angle of the line and the plane.**

A **figure** is said to be **projected on a plane** when all of its points are projected on the plane.

The plane determined by a line and its projection on another plane is called the **projecting plane.**

In the figure of prop. X, $\angle B'OB$ is the *inclination* of AB to MN. The plane determined by AB, $A'B'$ is the *projecting plane.*

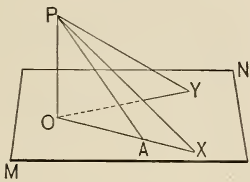
PROPOSITION XI.

346. Theorem. *Of all lines that can be drawn from a point to a plane,*

1. The perpendicular is the shortest;

2. Obliques with equal inclinations are equal, and conversely;

3. Obliques with equal projections are equal, and conversely.



1. Given $PO \perp$ plane MN, PX oblique to MN.

To prove that $PO < PX$.

Proof. $\because \angle XOP =$ rt. $\angle$, (Why?) $\therefore PO < PX$. I, prop. XX

2. Given $PO \perp MN, \angle PYO = \angle PXO$.

To prove that $PY = PX$, which is true because $\triangle POY \cong \triangle POX$. I, prop. XIX, cor. 7

CONVERSELY: **Given** $PO \perp MN$, $PY = PX$.

To prove that $\angle PYO = \angle PXO$, which is true because $\triangle POY \cong \triangle POX$. I, prop. XIX, cor. 5

3. Given $PO \perp MN$, $OY = OX$.

To prove that $PY = PX$, which is true because $\triangle POY \cong \triangle POX$. I, prop. I

CONVERSELY: **Given** $PO \perp MN$, $PY = PX$.

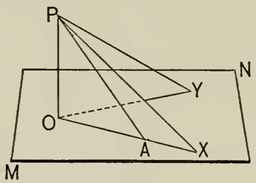
To prove that $OY = OX$, which is true because $\triangle POY \cong \triangle POX$. I, prop. XIX, cor. 5

Proposition XII.

347. Theorem. *From a point to a plane,*

1. Of two obliques with unequal inclinations, that having the greater inclination is the shorter, and conversely;

2. Of two obliques with unequal projections, that having the longer projection is the longer, and conversely.



1. Given $PO \perp MN$, PY and PA two obliques such that $\angle PAO > \angle PYO$.

To prove that $PA < PY$.

Proof. 1. Suppose X taken on OA so that $OX = OY$.

2. Then $\triangle POX \cong \triangle POY$, and $PX = PY$. Why?

3. But PX, and $\therefore$ its equal PY, $> PA$. I, prop. XX

Conversely:

Given $PO \perp MN$, PY and PA two obliques such that $PA < PY$.

To prove that $\angle PAO > \angle PYO$.

Proof. 1. Suppose X taken on OA so that $OX = OY$.

2. Then $\triangle POX \cong \triangle POY$, $\angle PXO = \angle PYO$, and $PX = PY$. Why?

3. $\therefore PA < PX$, $\because PA < PY$. Given

4. X cannot fall on A, for then $PA \equiv PX$.

5. Nor between O and A, for then $PA > PX$. Why?

6. $\therefore X$ is on OA produced,

and $\therefore \angle PAO > \angle PXO$. I, prop. V

7. $\therefore \angle PAO > \angle PYO$. Subst.

2. Given $PO \perp MN$, PY and PA two obliques such that $OA < OY$.

To prove that $PA < PY$.

Proof. 1. Suppose X taken on OA so that $OX = OY$.

2. Then $\triangle POX \cong \triangle POY$, and $PX = PY$. I, prop. I

3. And $\because OA < OY$, or OX,

$\therefore PA < PX$, or PY. Why?

CONVERSELY:

Given $PO \perp MN$, PY and PA two obliques such that $PA < PY$.

To prove that $OA < OY$.

Proof left for the student.

Definition. The length of the perpendicular from a point to a plane is called the **distance** from that point to the plane.

E.g. in the figure on p. 258, the distance from P to MN is the length of PO.

Exercises. 539. Prove that if three concurrent lines meet a fourth line, not in the same point, the four lines are coplanar.

540. Why does folding a sheet of paper give a straight edge?

541. Suppose it known that a point P is in each of the three planes X, Y, Z. Is P probably fixed? Is it necessarily fixed?

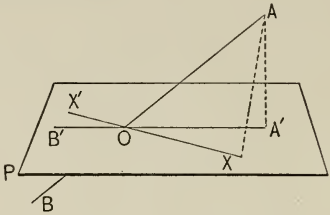
542. If the triangles ABC, $A'B'C'$, in different planes, are such that AB and $A'B'$ meet when produced, as also BC and $B'C'$, and CA and $C'A'$, then the lines AA', BB', CC' are either concurrent or parallel.

543. How many planes are determined by n concurrent lines, no three of which are coplanar?

544. If a line cuts one of two parallel lines, must it cut the other? If it does, are the corresponding angles equal?

Proposition XIII.

348. Theorem. *The acute angle which a line makes with its own projection on a plane is the least angle which it makes with any line in that plane.*



Given the line AB, cutting plane P at O, $A'B'$ the projection of AB on P, and XX' any other line in P, through O.

To prove that $\angle A'OA < \angle XOA$.

Proof. 1. Suppose A' the projection of A, OX made equal to OA', and AX, AA' drawn.

2. Then $AA' < AX$. Prop. XI, 1

3. $\therefore$ in $\triangle$ OXA and $OA'A$, we have

$OX = OA'$, $OA \equiv OA$, and $AA' < AX$,

$\therefore \angle A'OA < \angle XOA$. I, prop. XI

Exercises. 545. The obtuse angle which a line makes with its own projection (produced) on a plane is the greatest angle which it makes with any line in that plane.

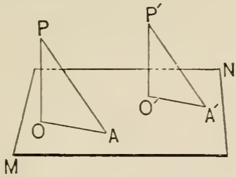
546. In a given plane, to determine the locus of points equidistant from two fixed points in space.

547. Prove prop. IX by supposing $O'Y'$ not perpendicular to MN, but supposing another line $O'Z$ from $O' \perp$ to MN; then prove that $O'Z$ would be parallel to OY, which would violate § 85, and hence be absurd.

548. As a special case of prop. X, suppose $AB \perp MN$; what would be its projection and its inclination?

PROPOSITION XIV.

349. Theorem. *Parallel lines intersecting the same plane are equally inclined to it.*



Given two parallels, PA, $P'A'$, intersecting a plane MN at A, A'; and O, O' the projections of P, P'.

To prove that $\angle PAO = \angle P'A'O'$.

Proof. 1. $\because PO$ and $P'O' \perp MN$,

$\therefore PO \parallel P'O'$. Why?

2. $\therefore \angle OPA = \angle O'P'A'$. Prop. V

(Let the student complete the proof.)

350. Definition. Two straight lines, not coplanar, are regarded as forming an **angle** which is equal to the one formed by either line and a line drawn, from a point upon it, parallel to the second.

E.g. in the figure of prop. XIV, the angle made by AO and $P'A'$ is considered as $\angle P'A'O'$ or $\angle PAO$.

Exercises. 549. Parallel line-segments are proportional to their projections on a plane.

550. In general, which is the longer, a line-segment or its projection? Is there any exception?

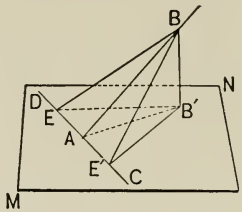
551. Show how, with a 10 ft. pole marked in feet, to determine the foot of the perpendicular let fall to the floor from the ceiling of a room 8 ft. high.

552. Show how a line 1 in. long and another 2 in. long may have equal projections on a plane.

553. If any two lines are parallel, respectively, to two others, an angle made by the first pair equals one made by the second.

Proposition XV.

351. Theorem. *If a line intersects a plane, the line in the plane perpendicular to the projection of the first line at the point of intersection is perpendicular to the line itself.*



Given AB intersecting the plane MN at A, B' the projection of B on MN, and $DC \perp AB'$ at A.

To prove that $DC \perp AB$.

Proof. 1. On DC let $EA = AE'$; join E, E' to B and B'.

2. Then $\triangle AB'E \cong \triangle AB'E'$, and $EB' = E'B'$. Why?

3. Then $\triangle EBB' \cong \triangle E'BB'$, and $EB = E'B$. Why?

4. Then $\triangle E'AB \cong \triangle EAB$, and $\angle E'AB = \angle BAE$. Why?

5. $\therefore DC \perp AB$, by defs. of rt. $\angle$ and $\perp$.

352. Definition. A line is said to be **parallel to a plane** when it never meets the plane, however far produced. In that case, also, the plane is said to be **parallel to the line.**

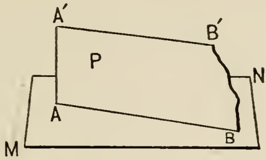
Exercises. 554. Prove prop. XV on the following outline: draw through B' a line $\parallel$ to DC; prove this parallel perpendicular to plane $AB'B$; $\therefore DC \perp$ plane $AB'B$, $\therefore DC \perp AB$.

555. Prove prop. XV by showing that $AE'^2 + AB^2 = BE'^2$, and that therefore $\angle E'AB$ is right.

556. In the figure of prop. XV prove that the area and perimeter of $\triangle AB'B$ are respectively less than those of $\triangle EB'B$.

PROPOSITION XVI.

353. Theorem. *Any plane containing only one of two parallel lines is parallel to the other.*



Given the parallel lines AB, $A'B'$, and the plane MN containing AB but not $A'B'$.

To prove that $MN \parallel A'B'$.

Proof. 1. AB and $A'B'$ determine a plane P. Prop. I, cor. 2

2. $\because$ AB and $A'B'$ lie wholly in P, $\therefore$ if $A'B'$ meets MN it meets AB. § 85

3. But $\because$ $AB \parallel A'B'$, this is impossible. § 82, def. $\parallel$ lines

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Exercises. 557. A line which is parallel to a plane is parallel to its projection on that plane.

558. Through a point without a straight line any number of planes can pass parallel to that line.

559. If a line is parallel to a plane, the intersection of that plane with any plane passing through that line is parallel to the line.

560. If from two points on a line parallel to a plane, parallel lines are drawn to and terminated by that plane, these parallel lines are equal.

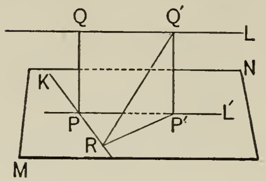
561. If a line is parallel to a plane, and if from any point in the plane a line is drawn parallel to the first line, then the second line lies wholly in the plane.

562. If, through a line parallel to a plane, several planes pass so as to intersect that plane, these lines of intersection are parallel.

563. If the distances from two given points on the same side of a plane, to that plane, are equal, the line determined by those points is parallel to the plane.

Proposition XVII.

354. Theorem. *Between two lines not in the same plane, one, and only one, common perpendicular can be drawn.*



Given two lines K, L, not coplanar.

To prove that one, and only one, common perpendicular can be drawn between them.

Proof. 1. Let MN be the plane, through K, $\parallel L$. (Can such a plane exist?) Let L' be the projection of L on MN.

2. Then K is not $\parallel$ to L', for then it would be $\parallel$ to L. Why?

Let K intersect L' at P.

3. A $\perp$ to L and K is $\perp$ to MN. Why?

4. Then $\because L'$ is the locus of the feet of all $\perp$s from points in L, on plane MN, § 343, def. projection

$\therefore P$ is the unique point in which a $\perp$ from a point on L, to K, can meet K.

5. $\therefore$ if PQ is drawn $\perp$ to L, it is $\perp$, and the only $\perp$, to both L and K.

Corollary. *The common perpendicular is the shortest line-segment between two lines not in the same plane.*

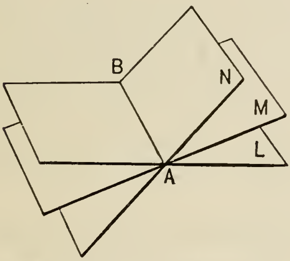
For if $Q'P' \parallel QP$, then $QP = Q'P' < Q'R$. Prop. XI, 1

355. Definition. The length of the common perpendicular from one line to another is called the **distance** between those lines.

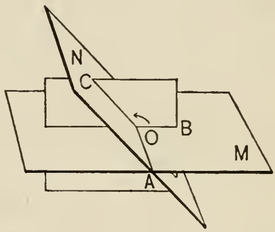
3. PENCIL OF PLANES.

356. Definitions. Any number of planes containing the same line are said to form a **pencil of planes**; the line is called its **axis.**

357. Any two planes of a pencil are said to form a **dihedral angle.**



LMN, a pencil of planes; *AB*, the axis of the pencil.



Dihedral angles formed by the planes *M* and *N*. Dihedral angle *MN* measured by plane angle *BOC*. *AO* the edge of the dihedral angle.

The two planes of a dihedral angle are called the **faces**, and the axis of the pencil is called the **edge** of the dihedral angle.

Two intersecting planes form more than one dihedral angle, just as two intersecting lines form more than one *plane angle*, the latter term now being used to designate an angle made by lines in a plane.

358. A plane of a pencil turning about the axis from one face of a dihedral angle to the other is said to turn through the angle, *the angle being greater* as the amount of turning is greater.

Since the size of a dihedral angle depends only upon the amount of turning just mentioned, it is *independent of the extent of the faces.*

359. If perpendiculars are erected from any point in the edge of a dihedral angle, one in each face, the size of the plane angle thus formed evidently varies as the size of the dihedral angle. Hence a dihedral angle is said to be *measured* by that plane angle, or, strictly, to have the same numerical measure.

360. A dihedral angle is said to be *acute, right, obtuse, oblique, reflex, straight,* according as the measuring plane angle is so, and it is usually named by its measuring plane angle, or merely by its faces in counter-clockwise order.

The terms *adjacent angles, bisector, sum and difference of dihedral angles, point within or without the angle, complement, supplement, conjugate,* and *vertical angles* will readily be understood from the corresponding terms in plane geometry.

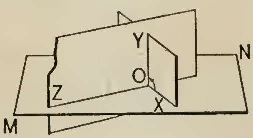
As with plane angles the smallest angle made by two intersecting lines is, in general, to be understood unless the contrary is stated, so with dihedral angles.

If a dihedral angle is right, the planes are said to be **perpendicular** to each other.

E.g. in the following figure, $ZY \perp MN$.

Proposition XVIII.

361. Theorem. *If a line is perpendicular to a plane, any plane containing this line is also perpendicular to that plane.*



Given OY perpendicular to the plane MN, and ZY any plane containing OY.

To prove that $ZY \perp MN$.

Proof. 1. Suppose OX, in MN, $\perp OZ$, the intersection of MN and ZY.

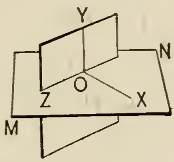
Then $\measuredangle$ YOZ, XOY are right $\measuredangle$. Why?

2. But $\because \angle XOY$ fixes the measure of the dihedral $\angle$, § 359

$\therefore ZY \perp MN$. Def.

PROPOSITION XIX.

362. Theorem. *If two planes are perpendicular to each other, any line in one of them, perpendicular to their intersection, is perpendicular to the other.*



Given the planes $ZY \perp MN$, OZ their intersection, and OX, in MN, $\perp OZ$.

To prove that $OX \perp ZY$.

Proof. 1. Let OY, in ZY, be $\perp$ to OZ at O.

Then $\angle XOY$ is the measuring angle. § 359

2. $\therefore \angle XOY$ is right. Def. $\perp$ planes

3. But $\because \angle ZOX$ is also right, Why?

$\therefore OX \perp ZY$. Why?

COROLLARIES. 1. *If two planes are perpendicular to each other, a line from any point in their line of intersection, perpendicular to either, lies in the other.*

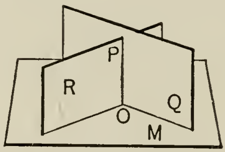
By the theorem, $OY \perp MN$, and it lies in ZY; and by prop. VII, cor. 2, only one perpendicular to MN can be drawn from O.

2. *Through a point without a line not more than one plane can pass perpendicular to that line.*

For if through Y another plane could pass $\perp OX$, it would pass through O, $\because \angle XOY =$ rt. $\angle$, and only one $\perp$ can be drawn from Y to OX. But the plane would also include line OZ, else there would be two $\perp$s from O to OX in the plane MN.

Proposition XX.

363. Theorem. *If each of two intersecting planes is perpendicular to a third plane, their line of intersection is also perpendicular to that plane.*



Given two planes, Q, R, intersecting in OP, and each perpendicular to plane M.

To prove that $OP \perp M$.

Proof. 1. A $\perp$ to M from O lies in Q and in R. Prop. XIX, cor. 1

2. $\therefore$ it coincides with OP, the only line common to Q and R. $\therefore OP \perp M$.

Exercises. 564. To construct a plane containing a given line, and parallel to another given line. (Assumed in step 1 of prop. XVII.)

565. Prove that vertical dihedral angles are equal.

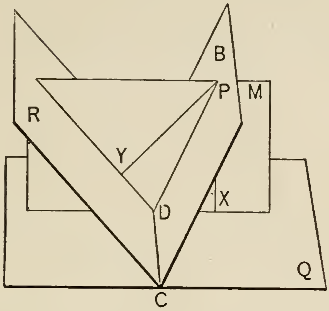
566. How many degrees in the measure of the dihedral angle between the plane of the earth's equator and the ecliptic?

567. Prove that the edge of a dihedral angle is perpendicular to the plane of the measuring angle.

568. Prove that a line and its projection on a plane determine a second plane perpendicular to the first.

PROPOSITION XXI.

364. Theorem. *Any point in a plane which bisects a dihedral angle is equidistant from the faces of the angle.*



Given a dihedral angle, with faces Q, R, and edge CD, bisected by plane B; P, any point in B, with $PX \perp Q$, $PY \perp R$.

To prove that $PX = PY$.

Proof. 1. Let M be the plane of PX, PY, and D its intersection with CD.

2. Then $\because PX \perp Q$, $\therefore M \perp Q$. Why?

3. Similarly, $M \perp R$. Why?

4. $\therefore M \perp CD$. Prop. XX

5. $\therefore CD \perp DX$, DY, DP, whose $\angle$s therefore measure the dihedral $\angle$s. § 359

6. $\therefore \angle XDP = \angle PDY$.

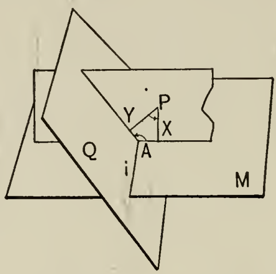
And $\because \angle X = \angle Y =$ rt. $\angle$, and $DP \equiv DP$,

7. $\therefore \triangle DXP \cong \triangle DYP$, and $PX = PY$. § 88, cor. 7

COROLLARY. *The locus of points that are equidistant from two intersecting planes is the pair of planes bisecting their dihedral angles.*

Proposition XXII.

365. Theorem. *If from any point lines are drawn perpendicular to two intersecting planes, the angle formed by these perpendiculars has a measure equal or supplemental to that of the dihedral angle of the planes.*



Given the planes M, Q, intersecting in i; lines $PX \perp M$, $PY \perp Q$; and plane PYX cutting i at A.

To prove that $\angle\ YPX$ is equal or supplemental to the dihedral angle MQ.

Proof. 1. Plane $YXP \perp M$, also $\perp Q$. Prop. XVIII

2. $\therefore$ plane $YXP \perp i$. Prop. XX

3. $\therefore XA$ and $YA \perp i$. § 339

4. $\therefore \angle\ XAY$ measures dihedral $\angle\ MQ$. § 359

5. But $\because \angle\ X = \angle\ Y =$ rt. $\angle$, Given

$\therefore \angle\ YPX$ is supplemental to $\angle\ XAY$, or dihedral $\angle\ MQ$. I, prop. XXI, cor.

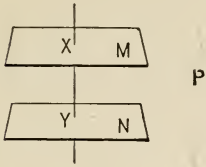
Corollary. *If the point is within the dihedral angle, the angles are supplemental.*

Definition. If two planes do not meet, however far produced, they are said to be **parallel**.

The term *pencil of parallels* is applied to planes as well as to lines.

PROPOSITION XXIII.

366. Theorem. *Planes perpendicular to the same straight line are parallel*



Given two planes, M, N, $\perp$ line XY, at X, Y, respectively.

To prove that $M \parallel N$.

Proof. If M and N should meet, as at P, then two planes would pass through $P \perp XY$, which is impossible.
Prop. XIX, cor. 2

Exercises. 569. Prove that through a point without a plane any number of lines can pass parallel to the plane.

570. Problem: To bisect a dihedral angle.

571. To find the locus of points equidistant from two fixed planes, and equidistant from two fixed points.

572. To find a point equidistant from two given planes, equidistant from two given points, and also at a given distance from a third plane.

573. Prove prop. XXII for the case in which the point P is taken in plane M.

574. In the figure on p. 270, as $\angle XAY$ increases from zero to a straight angle, what change does $\angle YPX$ undergo?

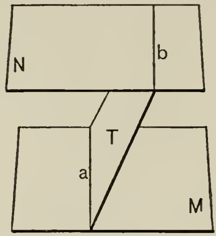
575. Also, suppose $\angle XAY = 120°$; what angle will PY make with plane M, if produced through Q to M?

576. Given two points, V, W, in two intersecting planes, M, Q, respectively. Find Z in the line of intersection of M and Q, such that $VZ + ZW$ shall be a minimum.

577. If from two points on a line parallel to a plane, parallel lines are drawn to that plane, a parallelogram is formed.

Proposition XXIV.

367. Theorem. *The lines in which two parallel planes intersect a third plane are parallel.*



Given two parallel planes, M, N, intersected by a third plane, T, in lines a, b.

To prove that $a \parallel b$.

Proof. 1. a and b are in the same plane T.

2. And they cannot meet, because they are in M and N respectively, and $M \parallel N$.

3. $\therefore$ they are parallel by definition.

Corollary. *A line perpendicular to one of two parallel planes is perpendicular to the other.*

Pass two planes through that line and apply prop. XXIV and the def. of a plane $\perp$ to a line.

Exercises. 578. Through a given point only one plane can pass parallel to a given plane.

579. If two parallel planes intersect two other parallel planes, the four lines of intersection are parallel.

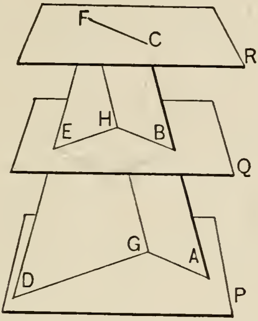
580. Parallel lines have parallel projections on any plane. (Suppose, as a special case, that the lines are perpendicular to the plane.)

581. If two lines are at right angles, are their projections on any plane also at right angles?

582. If two planes are perpendicular to each other, any line perpendicular to one of them is either parallel to or lies in the other.

PROPOSITION XXV.

368. Theorem. *If two straight lines are cut by parallel planes, the corresponding segments are proportional.*



Given ABC, DEF, two lines cut by planes P, Q, R, in points A, B, C and D, E, F.

To prove that $AB : BC = DE : EF$.

Proof. 1. Suppose the line GHF, drawn through F, $\parallel ABC$, cutting P, Q at G, H, respectively.

Then AC, GF determine a plane; also DF, GF.

Prop. I, cors. 2, 1

2. $\therefore AG \parallel BH \parallel CF$, and $DG \parallel EH$. Why?

3. $\therefore AB = GH$, and $BC = HF$. I, prop. XXIV

4. But $\because GH : HF = DE : EF$, IV, prop. X, cor. 1

$\therefore AB : BC = DE : EF$. Subst. 3 in 4

Exercises. 583. In a gymnasium swimming tank the water is 5 ft. deep, and the ceiling is 9 ft. above the water; a pole 18 ft. long rests obliquely on the bottom of the tank and touches the ceiling. How much of the pole is in the water?

584. In the figure of prop. XXV, connect C and D, and prove the theorem without using the line FG.

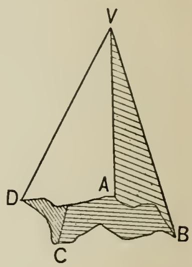
4. POLYHEDRAL ANGLES.

369. Definitions. When a portion of space is separated from the rest by three or more planes which meet in but one point, the planes are said to form, or to include, a **polyhedral angle.**

A polyhedral angle is also called a **solid angle.**

As two intersecting lines form an infinite number of plane angles, but the smallest is considered unless the contrary is stated, and similarly with two intersecting planes, so three or more intersecting planes form an infinite number of polyhedral angles, but, as with plane and dihedral angles, only the smallest is considered.

The lines of intersection of the planes of a polyhedral angle, each with the next, are called the **edges** of the polyhedral angle.



A polyhedral angle, $V\text{-}ABCD$. V, the vertex; VA, VB, VC, VD, the edges; planes VAB, VBC, ..., the faces.

On account of the complexity of the general figure, the planes which form a polyhedral angle are considered as cut off by the edges, as in the above figure. So also the edges, which may be produced indefinitely, are considered as cut off by the vertex unless the contrary is stated.

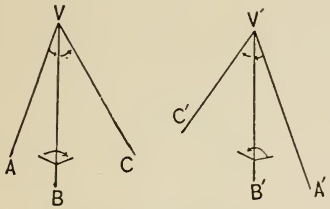
The portions of the planes which form a polyhedral angle, limited by the edges, are called the **faces** of the angle.

370. Polyhedral angles contained by 3, 4, ..., n planes are termed respectively **trihedral, tetrahedral, ... *n*-hedral** angles.

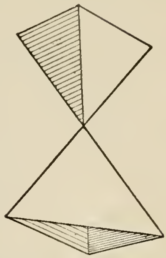
A polyhedral angle is specifically designated by a letter at its vertex, or by that letter followed by a hyphen, and letters on the successive edges.

371. Congruent polyhedral angles are such as have their dihedral angles equal, and the plane angles of their faces also equal and arranged in the same order.

372. Symmetric polyhedral angles are such as have their dihedral angles equal, and the plane angles of their faces also equal, but arranged in reverse order.



Symmetric polyhedral angles.



Opposite polyhedral angles.

Thus, in the above figure, V and V' are symmetric trihedral angles, the letters showing the reverse arrangement.

Some idea of this reverse arrangement may be obtained by thinking of two gloves, fitting the right and left hands respectively. As two such gloves are not congruent, so, in general, two symmetric polyhedral angles are not congruent.

373. Opposite polyhedral angles are such that each is formed by producing the edges and faces of the other through the vertex.

Exercises. 585. How many edges in an n-hedral angle? How many dihedral angles? How many plane face angles? How many vertices?

586. If a straight line is oblique to one of two parallel planes, it is to the other.

587. If a plane intersects all the faces of a tetrahedral angle, what kind of a plane figure is formed by the lines of intersection? What, in the case of a trihedral angle?

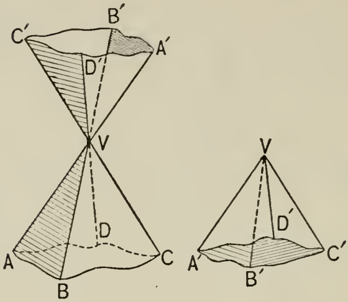
588. Does the magnitude of a polyhedral angle depend upon the lengths of the edges?

589. Construct from stiff paper two symmetric trihedral angles, with face angles of about 30°, 60°, 45°, and see if they are congruent. (No proof required.)

590. If each of two intersecting lines is parallel to a plane, so is the plane of those lines.

Proposition XXVI.

374. Theorem. *Opposite polyhedral angles are symmetric.*



Given $V\text{-}ABCD$, any polyhedral angle, and $V\text{-}A'B'C'D'$, its opposite polyhedral angle.

To prove that $V\text{-}ABCD$ and $V\text{-}A'B'C'D'$ are symmetric.

Proof. 1. $\angle AVB = \angle A'VB'$,

$\angle BVC = \angle B'VC'$, ... Prel. prop. VI

2. Dihedral $\angle$s with edges VB, VB', being formed by the same planes, have equal (vertical) measuring angles. Prel. prop. VI

3. So for the other dihedral $\angle$s. But the order of arrangement in the one is reversed in the other. $\therefore$ the polyhedral $\angle$s are symmetric.

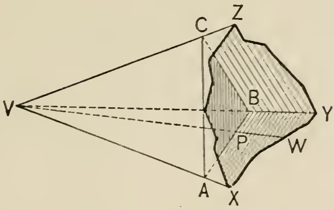
Note. That the order of the angles is reversed appears more clearly to the eye by making two opposite trihedral angles of pasteboard. It is also seen by tipping the upper angle over, as has been done in the figure to the right.

Exercises. 591. If the edges of one polyhedral angle are respectively perpendicular to the faces of a second polyhedral angle, then the edges of the latter are respectively perpendicular to the faces of the former.

592. Two parallel planes intersecting two parallel lines cut off equal segments.

PROPOSITION XXVII.

375. Theorem. *In any trihedral angle the sum of any two face-angles is greater than the third.*



Given the trihedral angle $V\text{-}XYZ$.

To prove that $\angle YVZ + \angle ZVX > \angle XVY$.

Proof. 1. If $\angle XVY \ngtr$ either $\angle YVZ$ or $\angle ZVX$, no proof is necessary. Why not?

2. If $\angle XVY >$ either $\angle YVZ$ or $\angle ZVX$, suppose it $> \angle ZVX$.

3. Then in plane VXY suppose VW drawn, making $\angle XVW = \angle ZVX$.

Suppose VC taken on VZ, equal to VP on VW, and a plane passed through C, P, and any point A of VX. Let this plane intersect VY at B.

4. Then $\triangle AVP \cong \triangle AVC$, and $AC = AP$. I, prop. I

5. But $\because AC + CB > AB$, or $AP + PB$, Why?

$\therefore CB > PB$. Why?

6. $\therefore$ in $\triangle$s PVB and CVB, $\angle BVC > \angle PVB$.
I, prop. XI

7. $\therefore \angle CVA + \angle BVC > \angle AVP + \angle PVB$, or $\angle AVB$.

Or $\angle YVZ + \angle ZVX > \angle XVY$. Ax. 4

Corollaries. 1. *In any trihedral angle the difference of any two face-angles is less than the third.*

For if the face-angles are a, b, c, then since $a + b > c$, $\therefore a > c - b$.

2. *In any polyhedral angle any face-angle is less than the sum of all the other face-angles.*

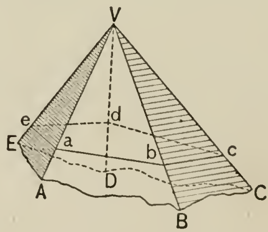
For the polyhedral angle may be divided into a number of trihedral angles, and prop. XXVII repeatedly applied.

Note. Prop. XXVII and corollaries suppose that each face-angle is less than a straight angle. This is in accordance with the note under the definition of a polyhedral angle.

376. Definition. A polyhedral angle is said to be **convex** when any polygon formed by a plane cutting every face, is convex; otherwise it is said to be **concave.**

Proposition XXVIII.

377. Theorem. *In any convex polyhedral angle the sum of the face-angles is less than a perigon.*



Given any convex polyhedral angle, $V\text{-}ABC$ .....

To prove that $\angle AVB + \angle BVC + \angle CVD + \cdots\cdots <$ perigon.

Proof. 1. Let the faces of the angle be cut by a plane. This will form a convex polygon of n sides ($n = 5$ in the figure), abc ..... Def. convex polyh. $\angle$

Let S_v = sum of plane $\measuredangle$ aVb, bVc, ..., at the vertex;

S_b = sum of plane $\measuredangle$ baV, Vba, cbV, ..., at the bases of the $\triangle$;

and S_p = sum of plane $\measuredangle$ cba, dcb, ..., of the polygon.

2. Then $S_p = (n-2)$ st. $\measuredangle$, or $S_p + 2$ st. $\measuredangle = n$ st. $\measuredangle$. I, prop. XXI

3. And $S_v + S_b = n$ st. $\measuredangle$, since there is a st. $\angle$ for each $\triangle$. I, prop. XIX

4. $\therefore S_v + S_b = S_p +$ perigon. Steps 2 and 3; ax. 1

5. And $\because S_b > S_p$, Prop. XXVII

$\therefore S_v <$ perigon.

Exercises. 593. The three planes which bisect the three dihedral angles of a trihedral angle intersect in a common line whose points are equidistant from the three faces. (See prop. XXI, cor., and I, prop. XLIV.)

594. Suppose a polyhedral angle formed by three, four, five equilateral triangles. What is the sum of the face-angles at the vertex?

595. If lines through any point O and the vertices, A, B, C, ..., of a polygon, cut a plane parallel to the plane of that polygon in A', B', C', ..., prove that $A'B'C'$ ..... $\sim ABC$ ..... and that the ratio of similitude is that of OA' to OA.

596. In ex. 595, the more remote O is from the planes ABC ....., $A'B'C'$ ....., the more nearly do AA', BB', CC' ..... become parallel; suppose they become parallel, state and prove the resulting theorem.

597. In ex. 595, if plane $A'B'C'$ ..... were not parallel to plane ABC ....., prove that the corresponding sides, AB, $A'B'$, and BC, $B'C'$, and CD, $C'D'$, ..., would, in general, meet in points on the intersection of the two planes.

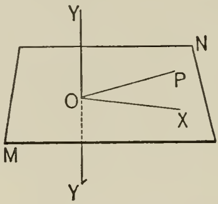
598. Two planes, each parallel to a third plane, are parallel to each other.

599. Ex. 598 is analogous to I, prop. XVIII. State the theorem and corollaries analogous to I, prop. XVII and its corollaries, and prove them.

5. PROBLEMS.

Proposition XXIX.

378. Problem. *Through a given point to pass a plane perpendicular to a given line: (1) the point being without the line, (2) the point being on the line.*



1. Given the line YY', and point P without.

Required through P to pass a plane $\perp$ YY'.

Construction. 1. From P draw $PO \perp YY'$. I, prop. XXX

2. From O draw another line $OX \perp YY'$. I, prop. XXIX

Then MN, the plane of OP, OX, is the required plane.

Proof. $\because YY' \perp OP$,

and $YY' \perp OX$,

$\therefore YY' \perp MN$. § 339

2. Given the line YY', and the point O upon it.

Required through O to pass a plane $\perp$ YY'.

Construction and Proof. Draw OP and $OX \perp YY'$. This can be done because the three lines are not required to be coplanar.

Then the plane $XOP \perp YY'$. § 339

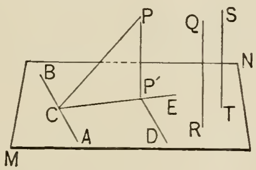
PROPOSITION XXX.

379. Problem. *Through a given point to pass a plane parallel to a given plane.*

Solution. Draw two intersecting lines in the given plane. Through the given point draw two lines parallel to these lines, thus determining the required plane.

PROPOSITION XXXI.

380. Problem. *Through a given point to draw a line perpendicular to a given plane: (1) the point being without the plane, (2) the point being in the plane.*



1. Given the plane MN, and the point P without.

Required to draw a perpendicular from P to MN.

Construction. 1. Draw $PC \perp AB$, any line in MN. I, prop. XXX

2. In MN draw $CE \perp AB$. I, prop. XXIX

3. Draw $PP' \perp CE$. I, prop. XXX

Then PP' is the required perpendicular.

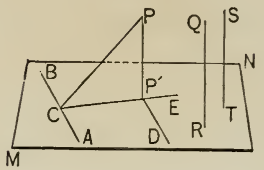
Proof. 1. $CA \perp$ plane CPP'. Prop. VI, cor. 1

2. Draw $P'D \parallel CA$; then $P'D \perp$ plane CPP'. Prop. IX

3. $\therefore \angle DP'P$ is right, and $PP' \perp P'D$. § 339

4. But $PP' \perp CP'$, and $\therefore PP' \perp MN$. Prop. VI, cor. 1

2. Given the plane MN, and the point R within it.



Required through R to draw a perpendicular to MN.

Construction. 1. From any external point S draw $ST \perp MN$. Case 1

2. From R draw $RQ \parallel TS$. I, prop. XXXIII

Proof. Then RQ is the required perpendicular. Why ?

Exercises. 600. From the point of intersection of two lines to draw a line perpendicular to each of them.

601. To determine the point whose distances from the three faces of a given trihedral angle are given. Is it unique ?

602. From the vertex of a trihedral angle to draw a line making equal angles with the three edges.

603. The three planes, through the bisectors of the face-angles of a trihedral angle, perpendicular to those faces, intersect in a common line whose points are equidistant from the edges. (See I, prop. XLIII.)

604. In how many ways can a polyhedral angle be formed with equilateral triangles and squares ?

605. Prove that a straight line makes equal angles with parallel planes.

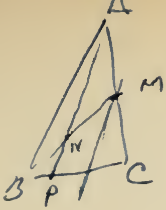
606. If each of two intersecting planes is parallel to a given line, prove that their intersection is coplanar with that line.

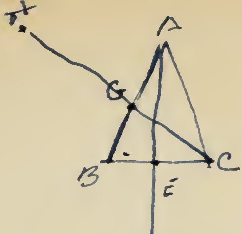
607. Prove that parallel lines make equal angles with parallel planes.

608. Are planes perpendicular to the same plane parallel ?

609. In the figure of prop. XXV, without drawing FG, draw CD and AF; then show that the four lines CD, CA, FD, FA intersect plane Q in the vertices of a parallelogram.

610. Given two lines, not coplanar, and a plane not containing either line, required to draw a straight line which shall cut both given lines and shall be perpendicular to the plane. (Project both lines on the plane.)





BOOK VII. — POLYHEDRA.

1. GENERAL AND REGULAR POLYHEDRA.

381. Definitions. A solid whose bounding surface consists entirely of planes is called a **polyhedron**; the polygons which bound it are called its **faces**; the sides of those polygons, its **edges**; and the points where the edges meet, its **vertices.**

382. If a polyhedron is such that no straight line can be drawn to cut its surface more than twice, it is said to be **convex**; otherwise it is said to be **concave.**

Unless the contrary is stated, the word *polyhedron* means convex polyhedron. The word *convex* will, however, be used wherever necessary for special emphasis.

383. If the faces of a polyhedron are congruent and regular polygons, and the polyhedral angles are all congruent, the polyhedron is said to be **regular.**

Exercises. 611. Draw a figure of a polyhedron of four faces. Count the edges, faces, and vertices, and show that the number of edges plus two equals the number of faces plus the number of vertices.

612. Do the same for a polyhedron of five faces; also of six faces.

613. Take a piece of chalk, apple, or potato, and see if a seven-edged polyhedron can be cut from it.

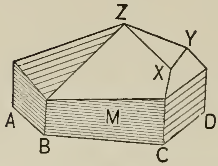
614. What is the locus of points on the surface of a polyhedron equidistant from two given vertices? (The distances are to be taken as usual on a straight line, and not necessarily on the surface.)

615. What is the locus of points equidistant from two given non-parallel faces of a given polyhedron?

616. To find a point equidistant from two given vertices of a polyhedron, and from two given non-parallel faces.

Proposition I.

384. Theorem. *If a convex polyhedron has* e *edges,* v *vertices, and* f *faces, then* e + 2 = f + v.



Given $ABC \ldots\ldots Z$, a convex polyhedron of e edges, v vertices, f faces.

To prove that $$e + 2 = f + v.$$

Proof. 1. Imagine $ABC \ldots\ldots Z$ formed by adding adjacent faces, beginning with any face as $ABCD \ldots\ldots$ of a sides, then adding face M, of b sides, and so on.

(It is advisable to build up a rectangular box of paste-board while reading the proof.)

Let $e_r =$ the number of edges, and $v_r =$ the number of vertices, after r faces have been put together.

(Thus when we put 2 rectangles together in building up the box, we have located 7 edges and 6 vertices; *i.e.* $e_2 = 7$, $v_2 = 6$, in this case.)

2. Then, since the first face had a sides, $\therefore e_1 = a$ and $v_1 = a$.

(In the box, $e_1 = 4$, $v_1 = 4$.)

3. $\because$ adding an adjacent face M, of b sides, gives only $(b-1)$ new edges, and $(b-2)$ new vertices (Why?),

(In the box, adding a second rectangle to the first gives only 3 new edges because we have 1 in common with the first face, and 2 new vertices because we have 2 in common with the first face.)

$\therefore e_2 = a + b - 1$, $v_2 = a + b - 2$, so that $e_2 - v_2 = 1$.

(In the box, $e_2 = 4 + 4 - 1 = 7$, $v_2 = 4 + 4 - 2 = 6$, so that $e_2 - v_2 = 1$.)

4. Therefore we have

$$e_2 = v_2 + 1.$$

Now while the number of edges common to two successive open surfaces will vary according to the way in which the additions are made, the addition of a new face evidently increases e by one more unit than it increases v.

$$\therefore e_3 = v_3 + 2,$$
$$e_4 = v_4 + 3,$$

and, in general,

$$e_r = v_r + r - 1,$$

or $$e_r - v_r = r - 1.$$

5. But the addition of the last, or fth face, as XYZ, after all the others have been put together, gives no new edges or vertices,

$$\therefore e_f - v_f = e_{f-1} - v_{f-1} = f - 2.$$

(In the box, adding the last face merely puts on the cover, adding no new edges or vertices. $\therefore e_6 - v_6 = e_5 - v_5 = 4$, which is evidently true, because 12, the number of edges, minus 8, the number of vertices, is 4.)

6. That is, $e - v = f - 2$, so that $e + 2 = f + v$; for $e_f = e$, and $v_f = v$.

Corollary. *For every polyhedron there is another which, with the same number of edges, has as many faces as the first has vertices, and as many vertices as the first has faces.*

It is easily seen that a polyhedron can be inscribed with a vertex at the center of each face, the number of edges remaining the same.

Note. This theorem is known as Euler's, although Descartes knew and employed it. The theorem is very useful in the study of crystals.

Exercise. 617. If the faces of a polyhedron are all triangular, the number of faces is even and is four less than twice the number of vertices. (Since there are 3 edges to every face, but each edge belongs to 2 adjacent faces, $e = 3f/2$; substitute in $e + 2 = f + v$.)

Proposition II.

385. Theorem. *There cannot be more than five regular convex polyhedra.*

Proof. 1. Let n = number of sides in one face, and a = number of degrees in each plane $\angle$ of the faces of a regular convex polyhedron.

Then $a = (n-2) \cdot 180°/n$, I, prop. XXI

and if $n = 6$, then $a = 120°$, and $3\,a = 360°$.

2. $\therefore$ if $n = 6$ or more, there can be no solid angle. VI, prop. XXVIII

3. And if $n = 5$, then $a = 108°$, and $3\,a = 324°$,

and $\therefore$ 3 regular pentagons, but no more, can form a solid angle. VI, prop. XXVIII

4. And if $n = 4$, then $a = 90°$, $3\,a = 270°$, $4\,a = 360°$,

and $\therefore$ 3 squares, but no more, can form a solid angle. VI, prop. XXVIII

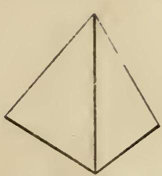
5. And if $n = 3$, $a = 60°$, $3\,a = 180°$, $4\,a = 240°$, $5\,a = 300°$, $6\,a = 360°$,

and $\therefore$ 3, 4, or 5 equilateral $\triangle$s, but no more, can form a solid angle. VI, prop. XXVIII

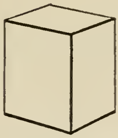
6. $\therefore$ there cannot be more than 5 regular convex polyhedra, viz. those formed by regular pentagons (3 at each vertex), squares (3 at each vertex), equilateral triangles (3, 4, or 5 at each vertex).

Note. There are five regular convex polyhedra; but the complete proof of the fact is not of enough importance to insert it in the body of the work. It may be given as an exercise, since it involves no new principles. These five polyhedra have been called the *Platonic Bodies*, from the attention given them in Plato's school, although they were known to the Pythagoreans. The three simpler forms enter largely into crystallography, usually somewhat modified.

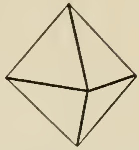
The five regular polyhedra are given on page 287.



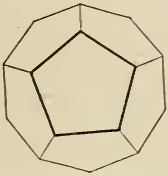
The regular *tetrahedron* (or triangular pyramid), formed by 4 equilateral triangles.



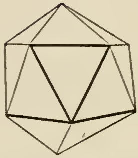
The regular *hexahedron* (or cube), formed by 6 squares.



The regular *octahedron*, formed by 8 equilateral triangles.

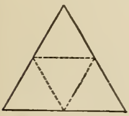


The regular *dodecahedron*, formed by 12 regular pentagons.

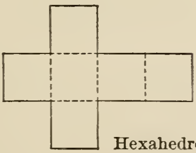


The regular *icosahedron*, formed by 20 equilateral triangles.

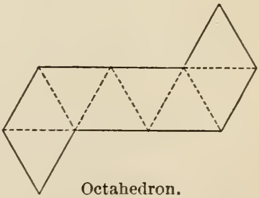
The five regular polyhedra can be constructed from cardboard by marking out the following, cutting through the heavy lines and half through the dotted ones, and then bringing the edges together.



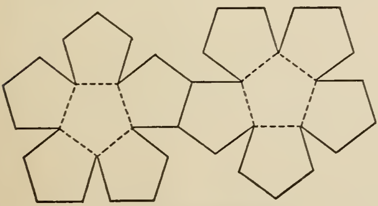
Tetrahedron.



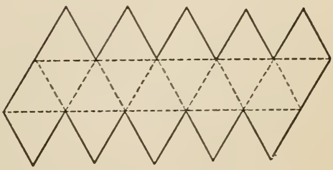
Hexahedron.



Octahedron.



Dodecahedron.



Icosahedron.

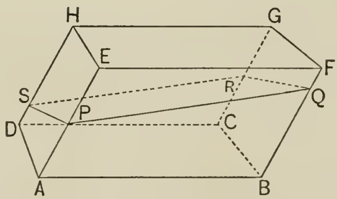
2. PARALLELEPIPEDS.

386. Definitions. A **Parallelepiped** is a solid bounded by three pairs of parallel planes.

The four lines through a parallelepiped, joining the opposite vertices, are called its **diagonals**.

Proposition III.

387. Theorem. *The opposite faces of a parallelepiped are congruent parallelograms; and any section of it, made by a plane cutting two pairs of opposite faces without cutting the remaining pair, is a parallelogram.*



Given the parallelepiped AG, and $PQRS$ a plane section cutting the parallel faces AF, DG, and AH, BG.

To prove (1) that AC and EG are congruent ▱s,
(2) that $PQRS$ is a ▱.

Proof. 1. $EF \parallel HG \parallel DC \parallel AB \parallel EF$,

and $BC \parallel FG \parallel EH \parallel AD \parallel BC$, VI, prop. XXIV

and $\therefore$ all faces are ▱s. § 97, def. ▱

2. $\therefore AB = EF = HG = DC$,

and $BC = FG = EH = AD$. I, prop. XXIV

3. And $\because \angle FEH = \angle BAD$, VI, prop. V

$\therefore \square AC \cong \square EG$, which proves (1). I, prop. XXVI

Similarly for other opposite faces.

4. And $\because PQ \parallel SR$, and $PS \parallel QR$, VI, prop. XXIV

$\therefore PR$ is a $\square$, which proves (2). § 97, def. $\square$

COROLLARY. *A parallelepiped has three sets of parallel and equal edges, four in each set.*

388. Definition. If the faces of a parallelepiped are all rectangles, it is called a **rectangular parallelepiped.**

It will be noticed that as axes of symmetry enter into the study of plane figures (§ 68), and especially of regular figures, so planes of symmetry and axes of symmetry enter into the study of solids. A *plane of symmetry* divides the solid into halves, related to each other as a figure is related to its image in a mirror. Planes of symmetry play an important part in the study of crystals. The term *axis of symmetry* will be understood from Plane Geometry.

Exercises. 618. Prepare a table showing the number (1) of faces, (2) of edges, (3) of vertices, (4) of sides in each face, (5) of plane angles at each vertex, of all of the five regular polyhedra.

619. How many degrees in the sum of the face-angles at one vertex of a regular tetrahedron? hexahedron? octahedron? dodecahedron? icosahedron?

620. The perpendiculars to the faces, through their centers, of a regular tetrahedron are concurrent in a point equidistant from all of the vertices, from all of the faces, and from all of the edges.

621. Prove that no polyhedron can have less than six edges.

622. In a regular tetrahedron three times the square on an altitude equals twice the square on an edge.

623. Certain crystals have their corners cut off, that is, the vertices of their polyhedral angles replaced by planes. Suppose a regular hexahedral crystal has its trihedral angles replaced by planes, how many faces has the new crystal? How many edges? vertices? Is Euler's theorem satisfied?

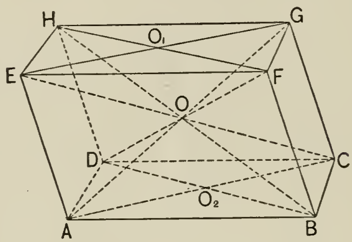
624. How many planes of symmetry and how many axes of symmetry has a regular hexahedron? octahedron?

PROPOSITION IV.

389. Theorem. *In any parallelepiped,*

1. The four diagonals are concurrent in the mid-point of each.

2. The sum of the squares on the four diagonals equals the sum of the squares on the twelve edges.



Given a parallelepiped with diagonals AG, BH, CE, DF.

To prove that (1) the diagonals are concurrent at O, the mid-point of each;

(2) $AG^2 + BH^2 + CE^2 + DF^2 = AB^2 + BC^2 + \cdots$

Proof. 1. $\because BF =$ and $\parallel DH$, Why?

$\therefore DBFH$ is a $\square$. Why?

2. $\therefore FD$ and BH bisect each other at O. § 100, cor. 2

Similarly, BH and CE, CE and AG, bisect each other.

3. And $\because$ there is only one point of bisection of BH and CE, § 41

$\therefore BH$, CE, AG, and DF are concurrent at O.

4. And $\because AG^2 + CE^2 = AC^2 + CG^2 + GE^2 + EA^2$,

and $DF^2 + BH^2 = BF^2 + FH^2 + HD^2 + DB^2$,

II, prop. XI, cor.

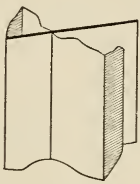
5. $\therefore$ by adding, and noting that $AC^2 + DB^2 = AB^2 + BC^2 + CD^2 + DA^2$, etc., the theorem is proved.

3. PRISMATIC AND PYRAMIDAL SPACE.
PRISMS AND PYRAMIDS.

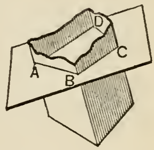
390. Definitions. A **prismatic surface** is a surface made up of portions of planes, the intersections of which are all parallel to one another.

391. If, counting from any plane of a prismatic surface as the first, each plane intersects its succeeding plane, and the last one intersects the first, the surface is said to enclose a **prismatic space.**

The lines of intersection are called the **edges,** and the portions of the planes between the edges, the **faces,** of the prismatic space.



A prismatic surface.



A portion of a prismatic space, quadrangular and convex. *ABCD*, a right section.

The edges and the faces are supposed to be unlimited in length. It will be readily seen that a prismatic space is related to entire space as a plane polygon is to its entire plane. It will therefore be inferred that theorems relating to polygons have corresponding theorems relating to prismatic spaces.

392. A section of a prismatic space, made by a plane cutting its edges, is called a **transverse section.** If it is perpendicular to the edges, it is called a **right section.**

393. A prismatic space is said to be **triangular, quadrangular, rectangular, pentagonal, ..., n-gonal,** according as a transverse section is a *triangle, quadrilateral, rectangle, pentagon, ...*, *n-gon,* and to be *convex* or *concave* according as a transverse section is a convex or a concave polygon.

Prismatic spaces may be such that transverse sections are convex, concave, or cross polygons. All theorems not involving mensuration will at once be seen to apply to each class. But on account of the complexity of the figures, the third form (cross) is not considered in this work.

394. The portion of a prismatic space included between two parallel transverse sections is called a **prism,** the two transverse sections being called the **bases** of the prism.

Thus in the figure on p. 293 the portion of the prismatic space P, between S and S', is a prism. S and S' are the bases.

The signification of the terms *edges*, *faces*, and *prismatic surface* of a *prism*, *upper* and *lower bases* of a *prism*, *triangular prisms*, etc., will be inferred from the above definitions. By transverse and right sections of a prism are to be understood the transverse and right sections of its prismatic space.

The sides of the *bases* of a *prism* are also called edges; where confusion is liable to arise these are called *base edges*, and the edges of the prismatic space are called *lateral edges*.

Exercises. 625. In the figure of prop. IV, prove that O_1, O, O_2 are collinear.

626. Also that $O_1O = EA/2$.

627. Also that if AG is a rectangular parallelepiped, O_1O is perpendicular to line EG.

628. Also that if the diagonals of all the faces are drawn, and the points of intersection of the diagonals of the opposite faces are connected, these connecting lines are concurrent at O, the mid-point of each.

629. Prove that the square on a diagonal of a rectangular parallelepiped equals the sum of the squares on three concurrent edges.

630. If the edge of a cube is represented by $\sqrt{3}$, find the diagonal.

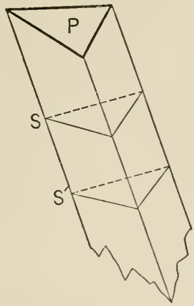
631. Prove that the four diagonals of a rectangular parallelepiped are equal.

632. Show that the edge, diagonal of a face, and diagonal, of a cube, are proportional to 1, $\sqrt{2}$, $\sqrt{3}$.

633. If the plane PR, of prop. III, were also to cut the faces HF and DB, what would be the plane figure resulting? What would be the relation of its opposite sides?

PROPOSITION V.

395. Theorem. *Parallel transverse sections of a prismatic space are congruent polygons.*



Given the prismatic space P, with S, S', two parallel transverse sections.

To prove that $S \cong S'$.

Proof. 1. $\because$ the sides of $S \parallel$ sides of S', respectively, VI, prop. XXIV

$\therefore$ ∡s of $S =$ ∡s of S', respectively. VI, prop. V

2. And $\because$ the sides of S also equal the sides of S', respectively, I, prop. XXIV

$\therefore$ by superposition, S is evidently congruent to S'.

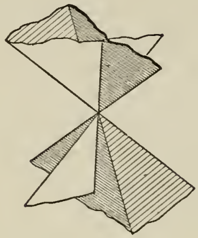
COROLLARIES. 1. *The bases of a prism are congruent polygons.*

2. *The faces of a prism are parallelograms.*

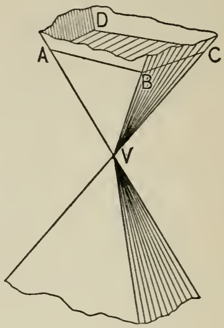
3. *The lateral edges of a prism are equal.*

Exercise. 634. Suppose in the figure of prop. III, another plane $\parallel$ to PR, cutting the same faces as PR, but not the other faces. Prove that it would cut out a parallelogram congruent to PR.

396. Definitions. A **pyramidal surface** is a surface made up of portions of planes which have but one point in common.



A pyramidal surface.



A portion of a pyramidal space, quadrangular and convex. V, the vertex; $ABCD$, a transverse section; $V\text{-}ABCD$, a pyramid, $ABCD$ being its base.

397. If, counting from any plane of a pyramidal surface as the first, each plane intersects its succeeding plane, and the last one intersects the first, the surface is said to contain a **pyramidal space.**

Unlike a prismatic space, a pyramidal space is double, its parts lying on opposite sides of the common point.

The lines of intersection of the planes are called the **edges,** the portions of the planes between the edges, the **faces,** and the point of intersection of the edges, the **vertex,** of the pyramidal space.

The edges and faces are supposed to be unlimited in length.

398. A section of a pyramidal space, made by a plane cutting all of its edges on the same side of the vertex, is called a **transverse section.**

399. The terms **triangular, ..., n-gonal, concave, convex** pyramidal space are defined as the like terms for prismatic space.

400. The portion of a pyramidal space included between the vertex and a transverse section is called a **pyramid,** the transverse section being called its **base,** and the vertex of the space, the **vertex** of the pyramid.

Thus the figure $V\text{-}ABC$ below is a pyramid, ABC being the base and V the vertex.

The distance from the vertex of a pyramid to the plane of its base is called the **altitude** of the pyramid.

Thus in the figure below, VV' is the altitude of pyramid $V\text{-}ABC$.

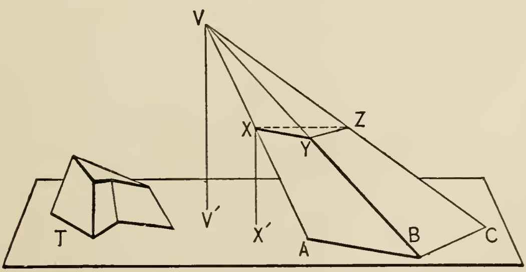
The signification of the terms *edges*, *faces*, *transverse section*, *base edges*, etc., of a pyramid can be inferred from the preceding definitions.

401. The portion of a pyramidal space included between two transverse sections on the same side of the vertex is called a **truncated pyramid**; if the transverse sections are parallel, it is called a **frustum** of a pyramid, the two sections being called the **bases** of the frustum.

A frustum of a pyramid is therefore a special form of a truncated pyramid.

A pyramid is also a special case of a truncated pyramid, the upper base being zero.

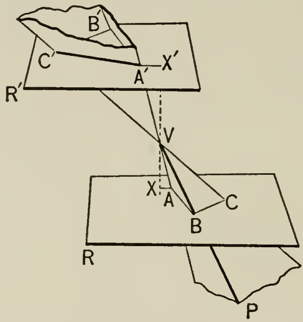
The distance from any point in one base of a frustum of a pyramid to the plane of the other base is called the **altitude** of the frustum.



T, a truncated pyramid; $ABCXYZ$, a frustum of the pyramid $V\text{-}ABC$; VV', the altitude of the pyramid; ABC, XYZ, the lower and upper bases of the frustum; XX', the altitude of the frustum.

Proposition VI.

402. Theorem. *Parallel transverse sections of a pyramidal space are similar polygons, whose areas are proportional to the squares of the distances from the vertex to the cutting planes.*



Given P, a pyramidal space with vertex V, cut by two parallel planes, R, R', making transverse sections $ABC \cdots\cdots = S$, $A'B'C' \cdots\cdots = S'$, respectively; $VX \perp R$, $VX' \perp R'$.

To prove that (1) $S \backsim S'$,

(2) $S : S' = VX^2 : VX'^2$.

Proof. 1. $\because$ the sides of S are $\parallel$ to the sides of S', VI, prop. XXIV

$\therefore VA : VA' = VB : VB' = \cdots\cdots$ and so on for other points. IV, prop. X, cor. 2

2. $\therefore S \backsim S'$, which proves (1). § 258, def. sim. figs.

3. And $\because AB : A'B' = VA : VA' = VX : VX'$, IV, prop. X, cors. 1, 3

and $S : S' = AB^2 : A'B'^2$, V, prop. IV

$\therefore S : S' = VX^2 : VX'^2$. IV, prop. VII

NOTE. The definition of similar figures, given in Book IV, § 258, is general; the center of similitude and the given figures may or may not be in the same plane. Thus in the figure on p. 296, V is the center of similitude of the triangles ABC and $A'B'C'$, and in the figure on p. 295, V is the center of similitude of the triangles XYZ and ABC. Neither is the definition limited to plane figures; we may have similar solids as well. Thus two balls are similar, or two cubes, or two regular tetrahedra, etc.

COROLLARIES. 1. *If a pyramid is cut by a plane parallel to the base,* (1) *the edges and altitude are divided proportionally,* (2) *the section is similar to the base.*

If the planes in the proof on p. 296 are on the same side of V, step 3 proves (1), and step 2 proves (2). Or, in the figure on p. 295,

$$VV' : XX' = VA : XA,$$

and $$\triangle ABC \backsim \triangle XYZ.$$

2. *In pyramids having equal bases and equal altitudes, transverse sections parallel to the bases, and equidistant from them, are equal; if the bases are congruent, so are the sections.*

1. Let s, s' be the areas of the sections, b, b' the areas of the bases, d the distance of the section from the vertex, and a the altitude.

2. Then from prop. VI,

$$s : b = d^2 : a^2,$$

and $$s' : b' = d^2 : a^2.$$

$$\therefore s : b = s' : b'. \qquad \text{Ax. 1}$$

3. But $$b = b',$$

$$\therefore s = s'.$$

4. And if the bases are congruent, so are the sections, since they are both similar and equal to the bases.

3. *The bases of a frustum of a pyramid are similar figures.*

For they are parallel transverse sections of a pyramidal space. Hence step 2, p. 296, proves the corollary.

Exercise. 635. In the figure on p. 296, suppose $\angle ABC$ a right angle, $AB = 3$ in., $AC = 5$ in., $VB = 10$ in., and the area of $\triangle A'B'C' = 12$ sq. in.; find the length of VB'.

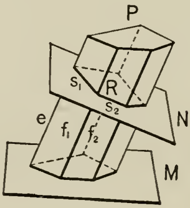
4. THE MENSURATION OF THE PRISM.

403. Definition. The area of the prismatic surface of a prism is called the **lateral area** of the prism.

Similarly for the pyramid, and for the cylinder and cone, to be defined hereafter.

Proposition VII.

404. Theorem. *The lateral area of a prism equals the product of an edge and the perimeter of a right section.*



Given the prism P; a right section R with sides s_1, s_2, ... ; f_1, f_2, ... the faces of the prism; e, an edge.

To prove that the lateral area of P is $e \cdot (s_1 + s_2 + \cdots\cdots)$.

Proof. 1. $\because$ by definition of right section, $R \perp e$, $\therefore s_1 \perp e$. § 339

2. $\because f_1, f_2, \ldots\ldots$ are ▱s, Prop. V, cor. 2

$\therefore$ area $f_1 = e \cdot s_1$. V, prop. II, cor. 3

3. And $\because$ the edges are equal, Prop. V, cor. 3

$\therefore$ area $f_2 = e \cdot s_2$, and so for the other faces.

4. $\therefore$ lateral area $= e \cdot s_1 + e \cdot s_2 + \cdots\cdots = e \cdot (s_1 + s_2 + \cdots\cdots)$. Ax. 2

405. Definitions. A prism whose edges are perpendicular to the base is called a **right prism**; if the edges are oblique to the base it is called an **oblique prism.**

E.g. on p. 301 R is a right prism and O is an oblique prism. So a cube is a special kind of a right prism, and the parallelepiped illustrated on p. 290 is an oblique prism.

The distance from any point in one base of a prism to the plane of the other base is called the **altitude** of the prism.

Similarly for a parallelepiped, which is a special kind of prism.

COROLLARY. *The lateral area of a right prism equals the product of the altitude and the perimeter of the base.*

For the altitude here equals the edge.

Exercises. 636. Required the lateral area of a prism of edge 3 in., the right section being an equilateral triangle of area $\frac{1}{4}\sqrt{3}$ sq. in. Also the lateral area of one of edge 3 in., the right section being a square of diagonal $\sqrt{2}$ in.

637. Required the lateral area of a right prism whose base is a square of area 9 sq. in., and whose altitude equals the diagonal of the base. Also required the total area.

638. Required the total area of a right prism whose base is an equilateral triangle of area $\frac{1}{4}\sqrt{3}$, and whose altitude equals a base edge.

639. Required the total area of a right prism whose base is a regular hexagon whose side is 1 in., the altitude of the prism being equal to the diameter of the circle circumscribing the base.

640. Required the lateral area of a prism of edge $\frac{1}{6}$, the right section being a regular hexagon of area $\frac{3}{2}\sqrt{3}$.

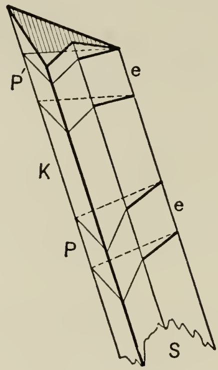
641. Required the total area of the prism mentioned in ex. 640, supposing it to be a right prism.

642. A converse of prop. VI is as follows: If two similar polygons have their corresponding sides parallel. and lie in different planes, the lines through their corresponding vertices are concurrent. Prove it. (A generalization of the idea of similar figures in perspective; see the definition of similar figures § 258, and the note at the top of p. 297.)

643. Investigate and prove whether or not any three faces of a tetrahedron are together greater than the fourth.

Proposition VIII.

406. Theorem. *Prisms cut from the same prismatic space and having equal edges are equal.*



Given two prisms, P, P', cut from the same prismatic space S, and having equal edges e.

To prove that $P = P'$.

Proof. 1. If $K =$ the portion of the prismatic space between P and P', then by adding e to the edges of K, each edge of $P + K =$ an edge of $K + P'$. Ax. 2

2. Then $\because P + K$ can evidently slide along in the prismatic space and occupy the position of $K + P'$,

$\therefore P + K \cong K + P'$. § 57

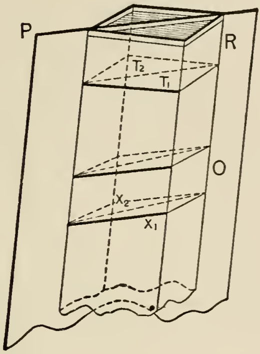
3. $\therefore P = P'$. Ax. 3

Corollaries. 1. *Right prisms having equal altitudes and congruent bases are congruent.*

2. *An oblique prism is equal to a right prism whose base and altitude are respectively a right section and edge of the oblique prism.*

PROPOSITION IX.

407. Theorem. *The two triangular prisms into which any parallelepiped is divided by a plane through two opposite edges are equal.*



Given O and R, parallelepipeds with equal edges, cut from a prismatic space, R being right; also, P, a plane through two opposite edges of that space, cutting R, O into two triangular prisms, T_1 and T_2, X_1 and X_2, respectively.

To prove that (1) $T_1 = T_2$, (2) $X_1 = X_2$.

Proof. 1. The base of $T_1 \cong$ the base of T_2. I, prop. XXIV

2. $\because$ they have the same altitude, $\therefore T_1 = T_2$. Why?

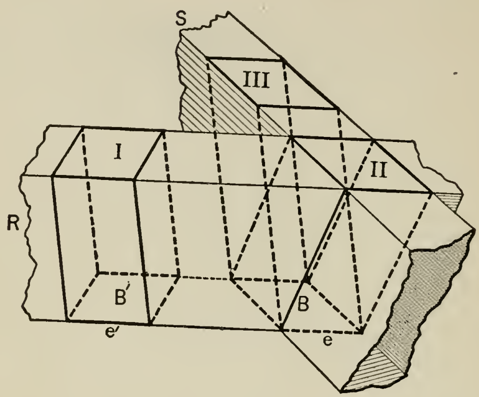
3. $X_1 = T_1$, and $X_2 = T_2$. Prop. VIII

4. And $\because T_1 = T_2$, $\therefore X_1 = X_2$. Ax. 1

COROLLARY. *A triangular prism is half of a parallelepiped of the same altitude, whose base is the parallelogram of which one side of the triangular base is the diagonal and the other two are the sides.* (Why?)

Proposition X.

408. Theorem. *Any parallelepiped is equal to a rectangular parallelepiped of equal base and equal altitude.*



Given a parallelepiped, III.

To prove that III equals a rectangular parallelepiped as I, of equal base and equal altitude.

Proof. 1. Let II be a parallelepiped on the same base, B, as III, formed by a rectangular prismatic space, R, cutting the prismatic space S of the figure.

Let I be a rectangular parallelepiped cut from R, with a base $B' = B$, and a base edge e' consequently equal to base edge e of II. II, prop. I, cor. 4

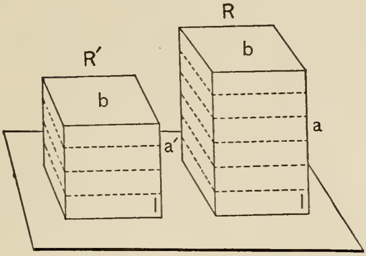
2. Then III = II, being part of S and having a common edge. Prop. VIII

3. And I = II, being part of R and having an equal edge. Prop. VIII

4. $\therefore$ III = I. Ax. 1

Proposition XI.

409. Theorem. *Two rectangular parallelepipeds having congruent bases are proportional to their altitudes.*



Given two rectangular parallelepipeds, R and R', with altitudes a and a' respectively, and with bases b.

To prove that $R : R' = a : a'$.

Proof. 1. Suppose a and a' divided into equal segments, l, and suppose $a = nl$, and $a' = n'l$.

(In the figures, $n = 6$, $n' = 4$.)

Then if planes pass through the points of division, parallel to the bases,

$R = n$ congruent rectangular ppds. bl,

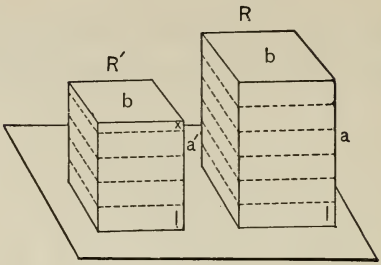
and $R' = n'$ " " " "

2. $\therefore \dfrac{R}{R'} = \dfrac{n \cdot bl}{n' \cdot bl} = \dfrac{n}{n'} = \dfrac{a}{a'}$. Why?

Note. The student should notice the resemblance between this theorem and Bk. V, prop. X. The above proof assumes that a and a' are commensurable, and hence that they can be divided into equal segments l. The proposition is, however, entirely general. The proof on p. 304 is valid if a and a' are incommensurable.

Exercise. 644. Given the diagonals, a, b, c, of three unequal faces of a rectangular parallelepiped, to compute the edges.

410. Proof for incommensurable case.



1. Suppose a divided into equal segments l, and suppose $a = nl$, while $a' = n'l +$ some remainder x, such that $x < l$.

   Then if planes pass through the points of division, parallel to the bases,

   $R = n$ congruent rectangular ppds. bl,

   and $R' = n'$ " " " " + a remainder bx such that $bx < bl$.

2. Then a' lies between $n'l$ and $(n'+1)\,l$, Why?

   and R' lies between $n' \cdot bl$ and $(n'+1)\,bl$. Why?

3. $\therefore \dfrac{a'}{a}$ lies between $\dfrac{n'}{n}$ and $\dfrac{n'+1}{n}$,

   while $\dfrac{R'}{R}$ lies between $\dfrac{n'}{n}$ and $\dfrac{n'+1}{n}$.

4. $\therefore \dfrac{a'}{a}$ and $\dfrac{R'}{R}$ differ by less than $\dfrac{1}{n}$. Why?

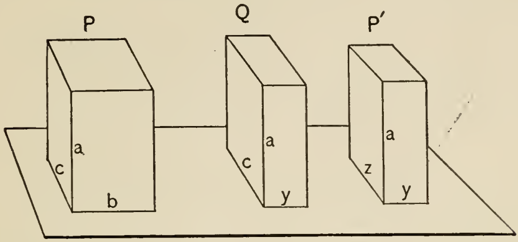
5. And $\because \dfrac{1}{n}$ can be made smaller than any assumed difference, by increasing n,

   $\therefore$ to assume any difference leads to an absurdity.

6. $\therefore \dfrac{a'}{a} = \dfrac{R'}{R}$, whence $\dfrac{R}{R'} = \dfrac{a}{a'}$.

PROPOSITION XII.

411. Theorem. *Two rectangular parallelepipeds of equal altitudes are proportional to their bases.*



Given two rectangular parallelepipeds, P, P', having altitudes a, and bases bc and yz, respectively.

To prove that $P : P' = bc : yz.$

Proof. 1. Suppose a rectangular parallelepiped Q to have an altitude a and a base yc.

2. Then $\because$ ac may be considered the base of P and Q,

$$\therefore \frac{P}{Q} = \frac{b}{y}. \qquad \text{Prop. XI}$$

3. And similarly, $\frac{Q}{P'} = \frac{c}{z}.$ Why ?

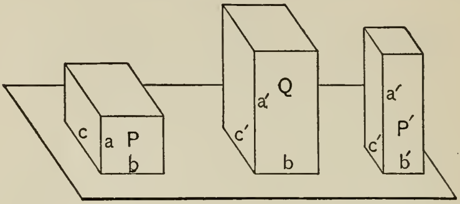
4. $$\therefore \frac{P}{P'} = \frac{bc}{yz}. \qquad \text{Why ?}$$

412. Definition. The length, breadth, and thickness of a rectangular parallelepiped are called its **three dimensions**.

Exercise. 645. If through a point on a diagonal plane of a parallelepiped planes are passed parallel to the two pairs of faces not intersected by the diagonal, the parallelepipeds on opposite sides of that diagonal plane are equal. (See II, prop. IV.)

Proposition XIII.

413. Theorem. *Two rectangular parallelepipeds are proportional to the products of their three dimensions.*



Given two rectangular parallelepipeds, P, P', of dimensions a, b, c, and a', b', c', respectively.

To prove that $P : P' = abc : a'b'c'$.

Proof. 1. Suppose a rectangular parallelepiped Q to have the three dimensions a', b, c'.

2. Then $$\frac{P}{Q} = \frac{ac}{a'c'},$$ Prop. XII

and $$\frac{Q}{P'} = \frac{b}{b'}.$$ Prop. XI

3. $$\therefore \frac{P}{P'} = \frac{abc}{a'b'c'}.$$ Why?

Corollaries. 1. *The volume of a rectangular parallelepiped equals the product of its three dimensions.*

This means that the *number* which represents the volume is the product of the three *numbers* representing the dimensions. That is, the *number* of times the unit of volume is contained in the given parallelepiped, is the product of the *numbers* of times the unit of length is contained in three concurrent edges.

If P' is a cube, of edges 1, 1, 1; then P' is the unit of measure of volume. But $P : P'$ then becomes $P : 1$, and $abc : 1 \cdot 1 \cdot 1$ then becomes $abc : 1$. $\therefore P : 1 = abc : 1$, or $P = abc$.

2. *The volume of any parallelepiped equals the product of its base and altitude.*

For (prop. X) it equals a rectangular parallelepiped of equal base and equal altitude.

3. *The volume of a triangular prism equals the product of its base and altitude.*

Cor. 2 with prop. IX, cor. Let the student give the proof in detail.

4. *The volume of any prism equals the product of its base and altitude.*

For it can be cut into triangular prisms by diagonal planes through a lateral edge, the sum of the bases of the triangular prisms being the base of the given prism. $\therefore$ cor. 3 applies. Let the student draw the figure and give the proof in detail.

5. *Any prism equals a rectangular parallelepiped of equal base and equal altitude.*

Cors. 4, 2.

6. *The volume of an oblique prism equals the product of an edge and a right section.*

Cor. 4 with prop. VIII, cor. 2.

7. *Prisms having equal bases are proportional to their altitudes.*

For if a is the altitude and b the base, then $P = ab$, and $P' = a'b'$. If $b = b'$, then $P' = a'b$. Hence $P : P' = ab : a'b = a : a'$.

8. *Prisms having equal altitudes are proportional to their bases. Prisms having equal bases and equal altitudes are equal.*

Let the student give the proof.

Exercises. 646. What is the edge of the cube whose volume equals that of a rectangular parallelepiped with edges 2.4 m, 0.9 m, 0.8 m?

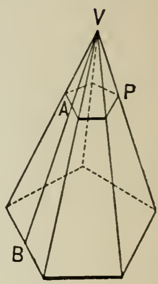
647. From the given edge e of a cube, compute (1) the cube's entire surface, (2) its diagonal, (3) its volume.

648. Draw a figure illustrating geometrically the formula

$$(a+b)^3 = a^3 + b^3 + 3\,a^2b + 3\,ab^2.$$

5. MENSURATION OF THE PYRAMID.

414. Definitions. A **regular pyramid** is a pyramid whose base is a regular convex polygon, the perpendicular to which, at its center, passes through the vertex of the pyramid.



415. The **slant height** of a *regular* pyramid is the distance from the vertex to any side of the base.

E.g. VB in the annexed figure.

416. The portion of the slant height of a *regular* pyramid cut off by the bases of a frustum is called the **slant height** of the **frustum.**

Corollary. *The slant height of a regular pyramid, or of a frustum of a regular pyramid, is the same on whatever face it is measured.*

Let the student show that the faces are all congruent; hence that the slant heights are equal.

Exercises. 649. To pass a plane through a given pyramid parallel to the base, so that the section shall equal half the base.

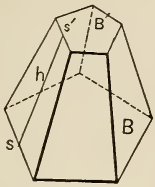
650. The edges of a rectangular parallelepiped are 3, 4, 5; required the total area of the faces, the areas of its diagonal planes, the length of its diagonal line, and the lengths of the diagonals of its faces. Similarly for a cube of edge $\sqrt{2}$.

651. If a cubic block of sandstone at a temperature of 0° Centigrade has an edge 1 m long, and if for every 1° Centigrade increase of temperature the edge increases 0.000012 of its length at 0°, find the volume at 40° Centigrade.

652. A brick has the dimensions 25 cm, 12 cm, 6 cm, but on account of shrinkage in baking, the mold is 27.5 cm long, and proportionally wide and deep. What per cent does the volume of the brick decrease in baking?

PROPOSITION XIV.

417. Theorem. *The lateral area of the frustum of a regular pyramid equals half the product of its slant height and the sum of the perimeters of its bases.*



Given BB', a frustum of a regular pyramid, h its slant height, s a side of base B and p its perimeter, s' a side of base B' and p' its perimeter, l the lateral area.

To prove that $l = \frac{1}{2} h (p + p')$.

Proof. 1. The area of each face $= \frac{1}{2} h (s + s')$.

V, prop. II, cor. 5

2. Adding all the faces, and remembering that p is the sum of the sides s, and p' of the sides s', we have $l = \frac{1}{2} h (p + p')$.

COROLLARY. *The lateral area of a regular pyramid equals half the product of its slant height and the perimeter of its base.*

For in the above theorem, let $B' = 0$; then s' and $p' = 0$; $\therefore l = \frac{1}{2} hp$.

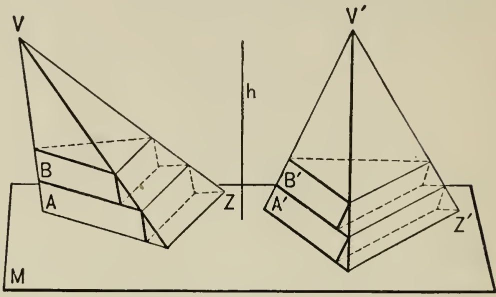
Exercises. 653. Prove the above corollary independently of the theorem.

654. What is the lateral area of a regular pyramid whose base is a triangle of altitude $\frac{a}{2}\sqrt{3}$, and whose slant height is a?

655. What is the total area of a frustum of a regular hexagonal pyramid whose base edges are respectively $3 - \sqrt{3}$ and $3 + \sqrt{3}$, and whose slant height is 10?

Proposition XV.

418. Theorem. *Pyramids having equal bases and equal altitudes are equal.*



Given pyramids VAZ, $V'A'Z'$, having equal bases, and having equal altitudes h.

To prove that pyramid $VAZ =$ pyramid $V'A'Z'$.

Proof. 1. Suppose their bases in the same plane M, and their vertices on the same side of M.

Suppose their altitude h divided into n equal parts and planes passed through the division-points parallel to M.

Then these planes will make equal corresponding transverse sections because the bases are equal.

Prop. VI, cor. 2

2. Suppose planes passed through the sides of these sections parallel to an edge of the pyramid, making a set of prisms in each pyramid, A, B, ... and A', B', ...

3. Then $\because A = A'$,

and $B = B'$, ... Prop. XIII, cor. 8

$\therefore A + B + \cdots\cdots = A' + B' + \cdots\cdots$ Ax. 2

4. But if n increases indefinitely,

$$A + B + \cdots\cdots \doteq \text{pyr. } VAZ,$$

and $$A' + B' + \cdots\cdots \doteq \text{pyr. } V'A'Z'.$$

5. $\therefore$ pyr. $VAZ =$ pyr. $V'A'Z'$. IV, prop. IX, cor. 1

COROLLARIES. 1. *A pyramid having a parallelogram for its base is divided into equal pyramids by a plane through its vertex and two opposite vertices of the base.*

For the two pyramids have equal bases and a common altitude.

2. *A pyramid having a parallelogram for its base equals twice a triangular pyramid of the same altitude, whose base equals half that parallelogram.* (Why?)

3. *A triangular pyramid can be constructed equal to any given n-gonal pyramid.*

II, prop. XII, and this theorem.

Exercises. 656. Find the area of the entire surface of a regular tetrahedron of altitude h.

657. Find the altitude of a regular tetrahedron of total area a.

658. Find, by § 417, the total area of a cube of edge e.

659. What is the length of the base edge of a regular triangular pyramid which is equal to a regular hexagonal pyramid of the same altitude, the base edge being 1?

660. In prop. XIV, cor., B' was supposed to decrease to 0; supposing, instead, that B' increases until it equals B, show that step 2 of the theorem gives the usual formula for the lateral area of a prism.

661. Prove that frustums of pyramids having equal bases and equal altitudes, which themselves have equal altitudes, are equal.

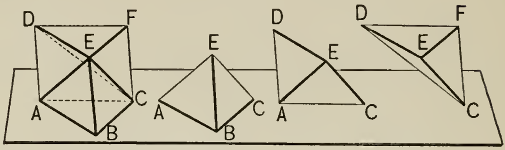
662. A pyramid has for its base a regular hexagon with its shorter diagonal $\sqrt{3}$; the altitude equals the longer diagonal; required the lateral area of the pyramid.

663. Find the total area of the pyramid mentioned in ex. 662.

664. The lower base of a frustum of a regular pyramid is a square of area s^2; the area of the upper base is half that of the lower one; the slant height is s; required the lateral area.

Proposition XVI.

419. Theorem. *A triangular prism can be divided into three equal triangular pyramids.*



Given $ABCDEF$, a triangular prism.

To prove that $ABCDEF$ can be divided into three equal triangular pyramids.

Proof. 1. $\because A, E, C$ determine a plane, also C, D, E, § 330, 1
$\therefore ABCDEF =$ three triangular pyramids, viz., $E\text{-}ABC$, $E\text{-}ACD$, $C\text{-}DEF$. Ax. 8

2. But $\because \triangle ABC \cong \triangle DEF$,
$\therefore E\text{-}ABC = C\text{-}DEF$. Prop. XV

3. And $C\text{-}DEF \equiv E\text{-}DCF = E\text{-}ACD$, because they have a common altitude from E to plane $ACFD$, and equal bases. Prop. XV

4. $\therefore E\text{-}ABC = C\text{-}DEF = E\text{-}ACD$. Ax. 1

Corollaries. 1. *A triangular pyramid is one-third of a triangular prism of the same base and same altitude.*

For the prism is three times the pyramid.

2. *Any pyramid is one-third of a prism of the same base and same altitude.*

For, dividing the base into triangles by drawing diagonals, the pyramid may be considered as made up of triangular pyramids, each of which is a third of a triangular prism of the same base and same altitude; hence the sum of the triangular pyramids, or the given pyramid, equals one-third the sum of the triangular prisms, or one-third of a prism of the same base and same altitude.

3. *The volume of a pyramid equals one-third the product of its base and altitude.*

Cor. 2, and prop. XIII, cor. 4.

4. *Pyramids having equal bases are proportional to their altitudes; having equal altitudes, to their bases.*

For if $p = \frac{1}{3}ab$, and $p' = \frac{1}{3}a'b'$, then $\dfrac{p}{p'} = \dfrac{\frac{1}{3}ab}{\frac{1}{3}a'b'} = \dfrac{ab}{a'b'}$.

And if $b = b'$, then $\dfrac{ab}{a'b'} = \dfrac{a}{a'}$.

Or if $a = a'$, then $\dfrac{ab}{a'b'} = \dfrac{b}{b'}$.

Or if $a = a'$ and $b = b'$, then $\dfrac{p}{p'} = 1$, or $p = p'$, as stated in prop. XV.

420. Definitions. A polyhedron which has for bases any two polygons in parallel planes, and for lateral faces triangles or trapezoids which have one side in common with one base and the opposite vertex or side in common with the other base, is called a **prismatoid.**

The **altitude** of a prismatoid is the perpendicular distance between the planes of its bases.

Exercises. 665. Find the volume of the pyramid mentioned in ex. 662.

666. A church-tower is capped by a regular octagonal pyramid whose height is 55.5 m, and whose base edge is 4.9 m. Required the volume.

667. A pentagonal pyramid has equal lateral and base edges, 1 in. Find the lateral area.

668. Find the volume of a cube the diagonal of whose face is $a\sqrt{2}$.

669. Each face of a given triangular pyramid is an equilateral triangle whose side is 3. Find the total area.

670. Find the volume of the tetrahedron mentioned in ex. 656.

671. Also of the pyramid mentioned in ex. 667.

672. An edge of a regular octahedron is 1 in. Find the volume.

673. A pyramid stands on a square base of edge 1 m; the lateral edge of the pyramid is also 1 m. Find the lateral area and volume.

Proposition XVII.

421. Theorem. *The volume of a prismatoid of bases* b *and* b′, *altitude* h, *and transverse section* m *midway between the bases, is expressed by the formula*

$$v = \frac{h}{6}(b + b' + 4m).$$

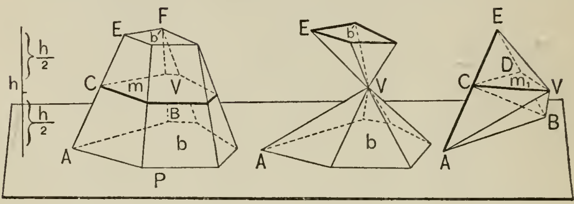


Fig. 1. Fig. 2. Fig. 3.

Proof. 1. If any face, $ABFE$, of the prismatoid P, is a trapezoid, divide it into two triangles by a diagonal EB. Let V be any point in m; join V to the vertices of P; then P will be divided into two pyramids (Fig. 2) of bases b, b' and vertex V, and also pyramids of vertex V and triangular bases ABE, etc. (Fig. 3.) Let EB meet m at D; call $\triangle\, VDC\ m_1$. (Fig. 3.)

2. Then the volume of

$$V\text{-}b = \tfrac{1}{3}\, b \cdot \frac{h}{2},$$

and $$V\text{-}b' = \tfrac{1}{3}\, b' \cdot \frac{h}{2}.$$ Prop. XVI, cor. 3

This completes Fig. 2.

3. Pyramid $V\text{-}ABE = E\text{-}CVD + B\text{-}CVD + V\text{-}ABC.$ Ax. 8

4. Of these, $$E\text{-}CVD = \tfrac{1}{3}\, m_1 \cdot \frac{h}{2},$$

and $$B\text{-}CVD = \tfrac{1}{3}\, m_1 \cdot \frac{h}{2}.$$ Prop. XVI, cor. 3

5. But $V\text{-}ABC = \text{twice } V\text{-}CBD$ (or $B\text{-}CVD$),

$\because \triangle ABC = \text{twice } \triangle CBD$,

having edge $AB = 2 \cdot CD$, and a common altitude.

Prop. XVI, cor. 4

6. $\therefore V\text{-}ABC = \frac{2}{3} m_1 \cdot \frac{h}{2}$.

7. $\therefore$ pyramid $V\text{-}ABE = \frac{h}{6} \cdot 4 m_1$, Axs. 2, 8

and $\therefore$ the sum of the pyramids of the form of

$V\text{-}ABE = \frac{h}{6} \cdot 4 m$. Axs. 2, 8

8. $\therefore P = \frac{h}{6}(b + b' + 4m)$. Axs. 2, 8

NOTE. The Prismatoid Formula, $v = \frac{h}{6}(b + b' + 4m)$, is of great value in the mensuration of solids. From it can be derived formulæ for the volumes of all of the solids of elementary geometry.

COROLLARY. *The volume of the frustum of a pyramid, of bases* b, b', *and altitude* h, *is* $\frac{h}{3}(b + b' + \sqrt{bb'})$.

For if e, e' are corresponding sides of b, b', then $\frac{1}{2}(e + e')$ is the corresponding side of m. (Why?)

$$\therefore \frac{e}{\frac{1}{2}(e+e')} = \frac{\sqrt{b}}{\sqrt{m}}, \text{ and } \frac{e'}{\frac{1}{2}(e+e')} = \frac{\sqrt{b'}}{\sqrt{m}}. \quad \text{(V, prop. IV.)}$$

$$\therefore \frac{e+e'}{\frac{1}{2}(e+e')} = \frac{\sqrt{b}+\sqrt{b'}}{\sqrt{m}}, \text{ and } \therefore 2\sqrt{m} = \sqrt{b} + \sqrt{b'}.$$

$\therefore 4m = b + b' + 2\sqrt{bb'}$, which may be substituted in the Prismatoid Formula.

Exercises. 674. By letting (1) $b' = 0$, and (2) $b' = b$, show that (1) prop. XVI, cor. 3, and (2) prop. XIII, cor. 4, follow, as special cases, from the Prismatoid Formula.

675. Calling a prismatoid whose lower base b is a rectangle of length l and width w, and whose upper base b' is a line e parallel to a base edge, and whose altitude is h, a *wedge*, find a formula for the volume of a wedge.

EXERCISES.

676. The base of a wedge is 4 by 6, the altitude is 5, and the edge e is 3. Find the volume. (See ex. 675.) Also, when $e = 0$.

677. The altitude of a pyramid is divided into five equal parts by planes parallel to the base. Find the ratios of the various frustums to one another and to the whole pyramid.

678. Two pyramids, P, P', have square bases, and are such that the altitude of P equals twice the altitude of P', but the base edge of P is half as long as the base edge of P'. Find the ratio of their volumes.

679. Find the volume of a cube whose diagonal is $\sqrt{3}$.

680. A frustum of a pyramid has for its bases squares whose sides are respectively 0.6 m, 0.5 m; the altitude of the frustum is 0.9 m. Find the volume.

681. Given the volume v, and the bases b, b', of a frustum of a pyramid, to find a formula for (1) its altitude, (2) the altitude of the whole pyramid.

682. A granite monument is in the form of a frustum of a square pyramid, surmounted by a pyramid; the sides of the bases of the frustum are 1 m and 0.8 m, and the altitude of the frustum is 1.8 m; the altitude of the pyramidal top is 0.45 m. A cubic meter of water weighs a metric ton, and granite is three times as heavy as water. Find the weight of the monument.

683. An excavation 1.5 m deep, rectangular at top and bottom, and in the form of a frustum of a pyramid, has its upper base 10 m wide and 16 m long, and the lower base 7.5 m wide. How many cubic meters of earth would it take to fill it to a depth of 0.75 m?

684. The volume of a cube is six times that of the regular octahedron formed by joining the centers of the faces of the cube.

685. Find the volume of a prismatoid of altitude 3.5 cm, the bases being rectangles whose corresponding dimensions are 3 cm by 2 cm, and 3.5 cm by 5 cm.

686. It is usual to find the volume of a pile of broken stones by taking the product of the altitude and the area of a transverse mid-section. Compare this with the Prismatoid Formula and find what relation it assumes between m and $b + b'$. Is this relation true in the case of a pyramid?

687. The volume of a pyramid equals the product of the altitude and a transverse section (parallel to the base) how far from the vertex?

BOOK VIII. — THE CYLINDER, CONE, AND SPHERE. SIMILAR SOLIDS.

1. THE CYLINDER.

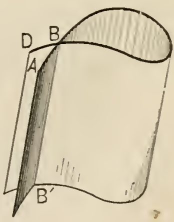
422. Definitions. A **curved surface** is a surface no part of which is plane.

The number of kinds of curved surfaces is unlimited, just as the number of kinds of curves in a plane is unlimited. But as among plane curves the circumference is the best known, so there are certain curved surfaces which are better known than others, and these are treated in this book.

423. A **cylindrical surface** is a surface generated by a straight line, called the **generatrix**, which moves so as constantly to pass through a given curve, called the **directrix**, and to remain parallel to its original position.

The surface of a piece of straight pipe, or the surface of the paper in a roll, is an example.

424. A straight line in any position of the generatrix is called an **element** of the cylindrical surface.



One form of a cylindrical surface. $AB\,CBD$, the directrix; BB', an element; BCB', a portion of a cylindrical space.

425. If the directrix is a closed curve, the cylindrical surface incloses a space of unlimited length, called a **cylindrical space**.

426. A section of a cylindrical space, made by a plane cutting its elements, is called a **transverse section**. If it is perpendicular to the elements, it is called a **right section**.

As a transverse section of a prismatic space may be a convex, concave, or cross polygon, so a transverse section of a cylindrical space may be a curve of any shape if only its end-points meet. All theorems, if the signs are properly considered, will be seen to apply to each of the three forms of transverse section, corresponding to convex, concave, and cross polygons. The third is, however, too complex for treatment in elementary works.

427. The portion of a cylindrical space included between two parallel transverse sections is called a **cylinder.**

E.g. the portion between planes P and P' in the figure on p. 319.

The terms *bases* and *altitude* of a cylinder will be understood, without further definition, from the corresponding definitions under the prism. The student should, throughout this section, notice the relation of cylindrical spaces to prismatic spaces.

A *cylinder* is considered as having the same *directrix* as its cylindrical space, and as having for *elements* the segments of the elements of the cylindrical surface included between its bases.

A cylinder is said to be **right** or **oblique** according as its elements are perpendicular or oblique to the bases.

If the base of a cylinder is a circle, the cylinder is said to be **circular.**

428. Postulate of the Cylinder. *A cylindrical surface may be constructed with any directrix and with any original position of the generatrix.*

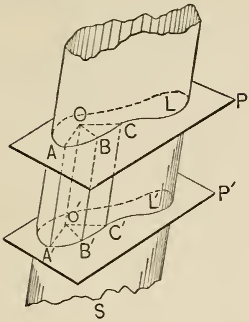
In solid geometry constructions are allowed which require other instruments than the compasses and straight-edge. For example, this postulate requires the generatrix to move constantly parallel to its original position, a construction manifestly impossible by the use of merely these two instruments.

Exercises. 688. Draw a figure of a convex cylinder; a concave cylinder; a cross cylinder.

689. Prove that if a transverse section of a cylindrical space is perpendicular to one element it is a right section.

PROPOSITION I.

429. Theorem. *Parallel transverse sections of a cylindrical space are congruent.*



Given a cylindrical space S, cut by two parallel planes, P, P', so as to form two transverse sections, L, L'.

To prove that $L \cong L'$.

Proof. 1. Let AA', BB', CC' be segments of elements between P and P', O any point in P, and $OO' \parallel AA'$, meeting P' at O'; join O to A, B, C, and O' to A', B', C'.

2. Then OO', AA' determine a plane. VI, prop. I, cor. 2

3. And $\because$ $OA \parallel O'A'$, $OB \parallel O'B'$, $OC \parallel O'C'$, § 367
$\therefore \angle AOB = \angle A'O'B'$, $\angle AOC = \angle A'O'C'$, ... § 337

4. Also, $OA = O'A'$, $OB = O'B'$, ... I, prop. XXIV
$\therefore$ if L is placed on L' so that O falls on O' and OA lies on $O'A'$, A will fall on A', B on B', etc.

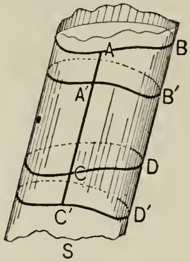
5. Similarly, for every point of L there is a single corresponding point of L' on which it will fall.
$\therefore$ the figures are congruent. § 57, def. congruence

COROLLARIES. 1. *The bases of a cylinder are congruent.*

2. *The elements of a cylinder are equal.* (Why?)

Proposition II.

430. Theorem. *Cylinders cut from the same cylindrical space, and having equal elements, are equal.*



Given two cylinders, AD, $A'D'$, cut from the same cylindrical space S, and having equal elements AC, $A'C'$.

To prove that $AD = A'D'$.

Proof. 1. $\because AC = A'C'$,

and $A'C \equiv A'C$,

$\therefore AA' = CC'$. Ax. 3

2. Similarly for BB' and DD', and for all other segments of the same elements, included between AB, $A'B'$, and CD, $C'D'$.

3. And $\because CD \cong AB$, and $C'D' \cong A'B'$, Prop. I

$\therefore$ solid CD' can be made to slide along in S and coincide with solid AB', since they are equal in all their parts.

4. Adding the common part $A'D$,

$$AD = A'D'. \qquad \text{Ax. 2}$$

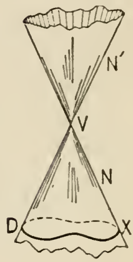
Corollary. *The cylindrical surfaces of two cylinders cut from the same cylindrical space, and having equal elements, are equal.*

For it is proved, in step 3, that they can be made to coincide.

2. THE CONE.

431. Definitions. A **conical surface** is a surface generated by a straight line which moves so as constantly to pass through a given curve and contain a given point called the **vertex.**

The terms *generatrix*, *directrix*, *elements* will be understood from §§ 423, 424.



A conical surface. DX, the directrix; V, the vertex; N, N', the lower and upper nappes; V-DX, a cone, with base the closed figure DX.

432. The portions of the conical surface on opposite sides of the vertex are called the **nappes,** and are usually distinguished as **upper** and **lower.**

433. If the directrix is a closed curve, the conical surface incloses a double space, on opposite sides of the vertex, known as a **conical space.**

A section of a conical space made by a plane cutting all of its elements on the same side of the vertex is called a **transverse section.**

434. The portion of a conical space included between the vertex and a transverse section is called a **cone,** the transverse section being called its **base.**

A *cone* is considered as having the same *directrix* and *vertex* as its conical space, and the segments of the elements between the vertex and base are called the *elements* of the cone.

The distance from the vertex of a cone to the plane of the base is called the **altitude** of the cone.

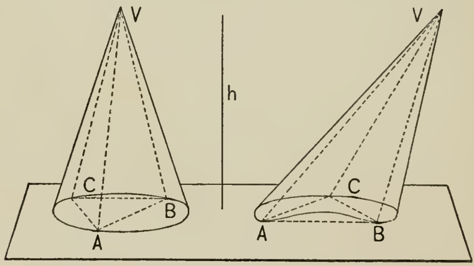
If the base of a cone is a circle, the cone is said to be **circular.** In that case the line determined by the vertex and the center of the base is called the **axis** of the cone. If this axis is perpendicular to the base, the cone is called a **right circular** cone; if oblique, an **oblique circular** cone.

A right circular cone is often called a *cone of revolution*, because it can be generated by the revolution of a right-angled triangle about one of its shorter sides. A right circular cylinder is often called a *cylinder of revolution.* (Why?)

435. Postulate of the Cone. *A conical surface may be constructed with any directrix and any vertex.*

436. Relation of Cone and Pyramid. If points A, B, C, ... are taken on the perimeter of the base of a cone, and joined to the vertex V, and if planes be passed through VA and VB, VB and VC, ..., a pyramid will be formed, called an **inscribed pyramid.**

If the base of the cone is bounded by a convex curve, the base of the pyramid will be a polygon inscribed in it. But whether the base is convex or not, the pyramid is called an *inscribed* pyramid.



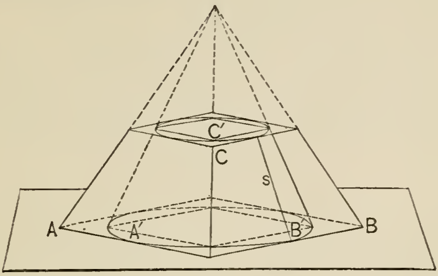
Pyramids inscribed in cones. The first figure is a right circular cone. The inscribed pyramids are indicated by dotted lines. h, the altitude.

437. If the base of the cone is a circle, and a regular polygon is circumscribed about it, the planes determined by the sides of the polygon and the vertex of the cone form, with the polygon, a pyramid which is said to be **circumscribed** about the circular cone.

There are other forms of circumscribed pyramids, but the one here mentioned is the only one that is necessary for this work.

The *slant height* of a right circular cone is defined as the slant height of the circumscribed pyramid. (Why?)

438. If a pyramid is inscribed in or circumscribed about a cone, a transverse section of the pyramid and cone cuts off, toward the base, a **frustum of a cone** and an **inscribed** or **circumscribed frustum of a pyramid.**



ABC, a circumscribed frustum of a pyramid; *A'B'C'*, an inscribed frustum of a regular pyramid; *s*, the slant height of the frustum of the cone.

The terms *bases*, *altitude*, and *lateral* surface will be understood from the terms used with the pyramid and the frustum of a pyramid.

439. From the above definitions it is evident that, if the inscribed or circumscribed frustum of a pyramid has equilateral bases, then if the number of lateral faces increases indefinitely, the frustum of the pyramid, its bases, and its lateral surface, approach as their respective limits the frustum of the cone, its bases, and its lateral surface, but that the altitude does not vary. If a frustum of a right pyramid be circumscribed about the frustum of a right circular cone, the slant height of the frustum of the pyramid may be called the *slant height of the frustum of the cone.* Hence the following

440. COROLLARY. *If* F *is the frustum of a cone, and* F' *the inscribed or circumscribed frustum of a pyramid, of equilateral bases, and if* b_1, b_2, l, v *are the bases, lateral surface, and volume, respectively, of* F, *and* b_1', b_2', l', v' *the bases, lateral surface, and volume, respectively, of* F', *then if the number of faces of* F' *increases indefinitely,*

$$b_1' \doteq b_1,\ \ b_2' \doteq b_2,\ \ l' \doteq l,\ \ v' \doteq v.$$

Proposition III.

441. Theorem. *The lateral area of a frustum of a right circular cone equals one-half the product of the slant height and the sum of the circumferences of its bases.*

Given a frustum of a right circular cone, l its lateral area, c_1 and c_2 the circumferences of its upper and lower bases, respectively, and s its slant height.

To prove that $l = \frac{1}{2} s(c_1 + c_2)$.

Proof. 1. Let l', p_1, p_2, s be the lateral area, the perimeters of the upper and lower bases, and the slant height, respectively, of the circumscribed frustum F of a regular pyramid.

2. Then $l' = \frac{1}{2} s(p_1 + p_2)$. VII, prop. XIV

3. But if the number of faces of F increases indefinitely, $l' \doteq l$, $p_1 \doteq c_1$, $p_2 \doteq c_2$, while the slant height is the same. § 440

4. $\therefore l = \frac{1}{2} s(c_1 + c_2)$. IV, prop. IX, cor. 1

Corollaries. 1. *If the radii of the upper and lower bases are* r_1, r_2, *respectively, then* $l = \pi s(r_1 + r_2)$.

2. *If* $r_3 =$ *the radius of the circle midway between the bases of the frustum, then* $l = 2\pi r_3 s$.

For $r_3 = (r_1 + r_2)/2$. Why?

3. *The lateral area of a right circular cone equals half the product of its slant height and the circumference of the base.*

If the upper base of a frustum of a cone decreases to zero, what does the frustum become? At the same time what does c_1 of step 4 become?

4. *The lateral area of a right circular cylinder equals the product of its altitude and the circumference of the base.*

If, in step 4, $c_1 = c_2$, what does l equal? What does s equal?

PROPOSITION IV.

442. Theorem. *The volume of the frustum of a cone of bases* b_1, b_2 *and altitude* h *is expressed by the formula*

$$v = \frac{h}{3}(b_1 + b_2 + \sqrt{b_1 b_2}).$$

Proof. 1. Let v', h, b_1', b_2' be the volume, altitude, and bases, respectively, of an inscribed frustum of a pyramid with an equilateral base.

2. Then $v' = \frac{h}{3}(b_1' + b_2' + \sqrt{b_1' b_2'})$. VII, prop. XVII

3. But if the number of faces of v' increases indefinitely, $v' \doteq v$, $b_1' \doteq b_1$, $b_2' \doteq b_2$, while h is constant. § 440

4. $\therefore v = \frac{h}{3}(b_1 + b_2 + \sqrt{b_1 b_2})$. IV, prop. IX, cor. 1

COROLLARIES. 1. *If the frustum is circular, and the radii of* b_1, b_2 *are* r_1, r_2, *respectively, then* $v = \frac{1}{3}\pi h (r_1^2 + r_2^2 + r_1 r_2)$.

2. *If* $r_3 =$ *the radius of the circle midway between the bases of a frustum of a circular cone, and if* h *is the altitude, and* r_1, r_2 *are the radii of the bases, then* $v = \frac{1}{6}\pi h (r_1^2 + r_2^2 + 4 r_3^2)$.

See prop. III, cor. 2.

3. *The volume of a cone of base* b *and altitude* h *is expressed by the formula* $v = \frac{1}{3} hb$.

Let $b_2 = 0$ in prop. IV.

4. *The volume of a circular cone, the radius of whose base is* r, *is expressed by the formula* $v = \frac{1}{3}\pi r^2 h$.

5. *The volume of a cylinder of base* b *and altitude* h *is expressed by the formula* $v = hb$.

Let $b_1 = b_2$.

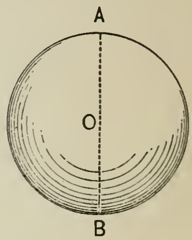
6. *The volume of a cylinder of altitude* h *and base radius* r *is expressed by the formula* $v = \pi r^2 h$.

3. THE SPHERE.

443. Definitions. A **sphere** is the finite portion of space bounded by a surface, which is called a **spherical surface** and is such that all points upon it are equidistant from a point within called the **center** of the sphere.

A straight line terminated by the center and the spherical surface is called a **radius,** and a straight line through the center, terminated both ways by the spherical surface, is called a **diameter** of the sphere.

A section of a sphere made by a plane is called a **plane section.**



A sphere. O, the center. OA, OB, radii. AB, a diameter.

444. Corollaries. 1. *A diameter of a sphere is equal to the sum of two radii of that sphere.*

2. *Spheres having the same radii are congruent, and conversely.*

3. *A point is within a sphere, on its surface, or outside the sphere, according as the distance from that point to the center is less than, equal to, or greater than, the radius.*

445. Postulates of the Sphere. (Compare § 109.)

1. *All radii of the same sphere are equal, and hence all diameters of the same sphere are equal.*

2. *If an unlimited straight line passes through a point within a sphere, it must cut the surface at least twice.*

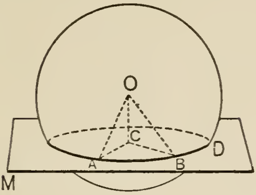
3. *If an unlimited plane, or if a spherical surface, intersects a spherical surface, it must intersect it in a closed line.*

4. *A sphere has but one center.*

5. *A sphere may be constructed with any center, and with a radius equal to any given line segment.*

PROPOSITION V.

446. Theorem. *A plane section of a sphere is a circle.*



Given a sphere with center O, and a section $ABDC$ made by a plane M.

To prove that $ABDC$ is a circle.

Proof. 1. M intersects the sphere in a closed line. § 445, 3

2. Suppose O joined to two points AB on that line, and $OC \perp M$; draw CA, CB.

3. Then $\because$ ∡ OCB, OCA are rt., and $OC \equiv OC$, and $OB = OA$,

$\therefore \triangle CBO \cong \triangle CAO$, and $CB = CA$. § 88, cor. 5

So for any other points on the closed line.

4. $\therefore$ $ABDC$ is a circle and C is its center. § 165, def. ⊙

447. Definitions. A *great circle* of a sphere is a circle passing through its center; a *small circle*, one not passing through its center.

COROLLARIES. 1. *The line determined by the center of a sphere and the center of any small circle of that sphere is perpendicular to that circle.*

For the line OC from the center of the sphere perpendicular to the circle has been proved to coincide with the line determined by the center of the circle and the center of the sphere, and there is only one line from the center of the sphere perpendicular to the circle.

2. *Of two circles of a sphere, the first is greater than, equal to, or less than, the second, according as its distance from the center is less than, equal to, or greater than, that of the second.*

For $AC^2 = r^2 - OC^2$; $\therefore$ the smaller OC, the greater AC, etc.

3. *A great circle has the same center and radius as the sphere itself; hence all great circles of a given sphere are equal.*

4. *A great circle bisects the sphere and the spherical surface.*

For if the two parts are applied one to the other, they will coincide; if they did not, the definition of sphere would be violated.

5. *Two great circles bisect each other.*

They have the same center, and hence a common diameter.

448. The student should notice the relation between the sphere and circle. Thus in prop. V and its corollaries:

The Circle.	The Sphere.
A portion of a *line* cut off by a *circumference* is a *chord*.	A portion of a *plane* cut off by a *spherical surface* is a *circle*.
The greater a *chord*, the less its distance from the center.	The greater a *circle*, the less its distance from the center.
A *diameter* (great chord) bisects the *circle* and the *circumference*.	A *great circle* bisects the *sphere* and the *spherical surface*.
Two *diameters* (great chords) bisect each other.	Two *great circles* bisect each other.

Hence may be anticipated a line of theorems on the sphere, derived from those on the circle, by making the following substitutions:

1. *Circle*, 2. *circumference*, 3. *line*, 4. *chord*, 5. *diameter*.	1. *Sphere*, 2. *spherical surface*, 3. *plane*, 4. *circle*, 5. *great circle*.

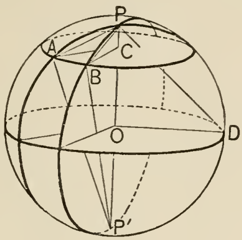
449. **Definitions.** The diameter of a sphere, perpendicular to a circle of that sphere, is called the **axis** of that circle, and its extremities are called the **poles** of that circle.

The two equal parts into which a great circle divides a sphere are called **hemispheres**, their curved surfaces being called **hemispherical surfaces**.

Corollary. *The axis of a circle passes through its center.*

PROPOSITION VI.

450. Theorem. *The straight lines joining any two points on the circumference of a circle of a sphere to one of the poles of that circle are equal.*



Given the circle ABC, and its poles P, P'; PA, PB connecting P with any two points on the circumference.

To prove that $PA = PB$.

Proof. 1. $\because OP \perp \odot ABC$ at C, $\therefore OP \perp AC$ and BC. Why?

2. And $\because PC \equiv PC$, and $CA = CB$, § 109

$\therefore \triangle ACP \cong \triangle BCP$, and $PA = PB$. Why?

COROLLARY. *Great-circle arcs from a pole of a circle to points on the circumference of that circle are equal.* (Why?)

451. Definitions. The length of the great-circle arc joining a pole to any point on the circumference of a circle is called the **polar distance** of the circle.

The shorter polar distance of small circles is to be understood.

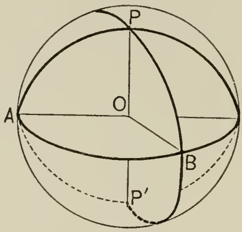
A fourth of the circumference of a great circle is called a **quadrant.**

COROLLARIES. 1. *Circles of the same sphere, having equal polar distances, are equal.* (Why?)

2. *The polar distance of a great circle is a quadrant.* (Why?)

Proposition VII.

452. Theorem. *If, on a spherical surface, each of the great-circle arcs joining a point to two other points (not the extremities of a diameter of the sphere) is a quadrant, then that point is a pole of the great circle through those points.*



Given P, A, B, three points on a spherical surface, and such that $\widehat{PA} = \widehat{PB} =$ a quadrant; A, B are not extremities of a diameter; O is the center.

To prove that P is the pole of the great circle ABO.

Proof. 1. $\because \widehat{PA} = \widehat{PB} =$ a quadrant,

$\therefore \angle POA = \angle POB =$ a rt. $\angle$. III, prop. II, cor. 2

2. $\therefore PO \perp \odot ABO$. VI, prop. VI, cor. 1

3. $\therefore P$ is a pole of $\odot ABO$. § 449, def. pole

Exercises. 690. How many points on a spherical surface determine a small circle? How many, in general, determine a great circle?

691. Prove that parallel circles of a sphere have the same poles.

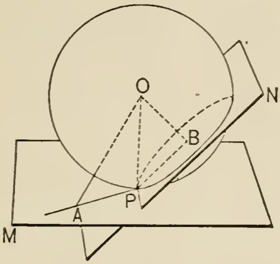
692. In the theorem: A diameter which is perpendicular to a chord bisects it, make the substitutions suggested in § 448, and prove the resulting proposition.

693. Similarly for III, prop. VI.

694. What is the locus of points at a given distance r from a fixed point C?

PROPOSITION VIII.

453. Theorem. *Of all planes through a point on a sphere the plane perpendicular to the radius drawn to that point is the only one that does not meet the sphere again.*



Given point P on a sphere with center O, and M, N, two planes respectively perpendicular and oblique to OP at P.

To prove that M does not meet the spherical surface again, but that N does.

Proof. 1. Let $OB \perp N$, and OA be any oblique to M.

Then $\because OP$ is oblique to N, Why?

$\therefore OB < OP$. VI, prop. XI

2. And $\because OP \perp M$,

$\therefore OA > OP$. VI, prop. XI

3. $\therefore B$ is within, and A without, the sphere. § 444, cor. 3

4. $\therefore N$ meets the spherical surface in more than one point. § 445, 3

5. And $\because A$ is any point in M, except P, Step 3

$\therefore M$ meets the surface only at P.

454. Definitions. A plane (or line) which, meeting a spherical surface in one point, does not meet it again, is said to be **tangent to the sphere** at that point. The point is called the **point of tangency**, or *point of contact*, and the plane (or line) is called a **tangent plane** (or **line**).

Corollaries. 1. *One and only one plane can be passed through a given point on a sphere, tangent to that sphere.* (Why?)

2. *Any tangent plane is perpendicular to the radius at the point of contact.*

For it cannot be oblique and be a tangent plane. Step 4.

3. *A plane perpendicular to a radius at its extremity on the spherical surface is tangent to the sphere.*

Exercises. 695. To find a point in a given plane, equidistant from two fixed points in that plane, and at a given distance d from a point C not in that plane. Discuss for 0, 1, 2 solutions.

696. Prove that the lateral area of any right cylinder equals the product of its altitude and the perimeter of the base. (Inscribe a prism and apply the theorem of limits.)

697. How many square feet in the surface of a cylindrical water tank, open at the top, its height being 40 ft., and its diameter 40 ft.?

698. Considering the moon as a circle of diameter 2160.6 miles whose center is 234,820 miles from the eye, what is the volume of the cone whose vertex is the eye and whose base is the full moon?

699. Find a point whose distance from a fixed point is d and whose distance from each of two intersecting planes is d'. Discuss the solution as to impossible cases, and the number of such points for possible cases.

700. Find the locus of points equidistant from two given points, and at a given distance d from a given point.

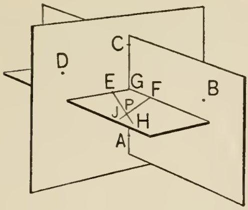
701. To determine a plane which shall pass

(1) through a given *line* and be at a given distance from a given *point*.

(2) through a given *point* and be at a given distance from a given *line*.

PROPOSITION IX.

455. Theorem. *Four points, not coplanar, determine a spherical surface.*



Given four points, A, B, C, D, not coplanar.

To prove that A, B, C, D determine a spherical surface.

Proof. 1. Draw AB, BC, CD, DA, AC.

Let E be the circumcenter of $\triangle ACD$, F of $\triangle ABC$, $EH \perp ACD$, $FJ \perp ABC$.

2. Then E, F are on the $\perp$ bisectors of AC; call these $\perp$ bisectors GE, GF. I, prop. XLI

3. And $\because EA = EC = ED$ (Why?),

$\therefore$ any point on EH is equidistant from A, C, D.

VI, prop. XI, 3

Similarly, any point on FJ is equidistant from A, B, C.

4. But $CA \perp$ plane EGF. Why?

5. $\therefore$ planes ABC, $ACD \perp EGF$. Why?

6. $\therefore$ both EH and FJ lie in plane EGF.

VI, prop. XIX, cor. 1

7. And $\because FJ$ meets EH, uniquely, as at P,

I, prop. XVII, cor. 4

$\therefore P$ is the center of a sphere whose surface passes through A, B, C, D, and there is only one such sphere.

456. Definitions. A sphere is said to be **circumscribed about a polyhedron** if the vertices of the polyhedron all lie on the spherical surface; the polyhedron is then said to be **inscribed in the sphere.**

COROLLARIES. 1. *Two spherical surfaces having four common points, not coplanar, coincide.*

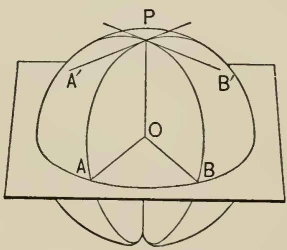
For by step 7 they have the same center, P, and the same radius.

2. *The perpendiculars to the four faces of a tetrahedron, through the circumcenters of those faces, are concurrent.*

For each of these perpendiculars passes through P, the center of the sphere whose surface is determined by the four vertices.

3. *A sphere can be circumscribed about any tetrahedron.*

457. The **angle between two great-circle arcs** is defined as being the plane angle between tangents to those arcs at their point of meeting.



E.g. the angle made by arcs AP, BP is defined as the plane angle $A'PB'$.

458. From this definition follow these *corollaries:*

1. *The angle made by two arcs has the same numerical measure as the dihedral angle of their planes.* (§ 359.)

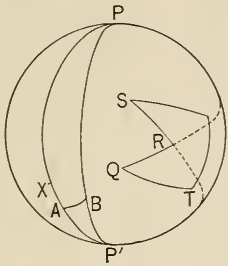
2. *An angle made by two arcs has the same numerical measure as the arc which these arcs intercept on the circumference of the great circle of which the vertex is the pole.*

That is, $\angle APB = \angle A'PB' = \angle AOB$, which has the same numerical measure as $\widehat{AB}$.

459. A **spherical polygon** is a portion of a spherical surface bounded by arcs of great circles.

The words *sides*, *angles*, *vertices*, etc., are used as with plane polygons.

460. A spherical polygon is said to be **convex** when each side produced leaves the entire polygon on the same hemisphere; otherwise it is said to be **concave.**



In the figure, ABP is a convex polygon, for if any side, as PB, is produced it leaves the entire polygon on the hemisphere to the left of PB. But $QRST$ is concave, because side SR, or QR, produced, leaves part of the polygon on one hemisphere thus formed, and part on the other.

461. COROLLARY. *No side of a convex spherical polygon is greater than a semicircumference.*

For if $AP >$ semicircumference, suppose $XP =$ a semicircumference. Then ∵ great circles bisect each other (prop. V, cor. 5), PB must pass through X; but then PB produced would leave part of the polygon on one hemisphere and part on the other, so that it could not be convex.

462. A **lune** is a portion of a spherical surface bounded by the semicircumferences of two great circles. The **angle of a lune** is that angle toward the lune made by the bounding arcs.

In the figure, $PAP'B$ is a lune, and $\angle APB$, or $\angle BP'A$, is its angle.

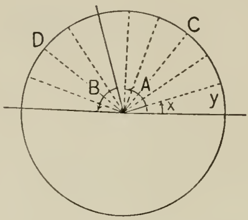
The limiting cases of a lune are evidently a semicircumference, when the angle is zero, and a spherical surface, when the angle is 360°.

463. COROLLARY. *Lunes on the same sphere, and having the same angle, are congruent.*

For one can evidently be made to coincide with the other.

Proposition X.

464. Theorem. *On the same sphere or on equal spheres lunes are proportional to their angles.*



(In this figure the eye is supposed to be looking down on the sphere from above the angle of the lune, as on the North Pole of the earth. This allows only half of each lune to be seen.)

Given two lunes, C and D, with angles A and B respectively.

To prove that $A : B = C : D.$

Proof. 1. If C and D are on different spheres, they can be placed in the relative positions shown in the figure. § 444, cor. 2

Suppose A and B divided into equal ∠s, x, and suppose $A = nx$, and $B = n'x$.

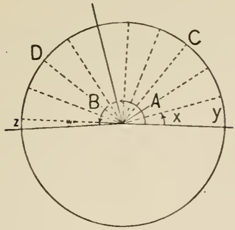
(In the figure $n = 6$, $n' = 4$.)

2. Then C is divided into n congruent lunes, y,
and D " " n' " " § 463

3. $$\therefore \frac{A}{B} \equiv \frac{nx}{n'x} = \frac{n}{n'} = \frac{ny}{n'y} \equiv \frac{C}{D}.$$ Why?

Exercise. 702. The six planes perpendicular to the six edges of a tetrahedron at the mid-points of its edges, meet in a point. (Is this point the center of a particular sphere?)

465. Proof for incommensurable case. (Compare § 410.)



1. Suppose A divided into equal $\angle$s, x, and suppose $A = nx$, while $B = n'x +$ some remainder w, such that $w < x$.

   Then C is divided into n congruent lunes, y, and D is the sum of n' congruent lunes, y, + a remainder z, such that $z < y$.

2. Then B lies between $n'x$ and $(n'+1)\,x$, Why?

   and D " " $n'y$ " $(n'+1)\,y$. Why?

3. $\therefore \dfrac{B}{A}$ lies between $\dfrac{n'x}{nx}$ and $\dfrac{(n'+1)\,x}{nx}$, Why?

   while $\dfrac{D}{C}$ lies between $\dfrac{n'y}{ny}$ and $\dfrac{(n'+1)\,y}{ny}$.

4. $\therefore \dfrac{B}{A}$ and $\dfrac{D}{C}$ both lie between $\dfrac{n'}{n}$ and $\dfrac{n'+1}{n}$.

5. $\therefore \dfrac{B}{A}$ and $\dfrac{D}{C}$ differ by less than $\dfrac{1}{n}$. Why?

6. And $\because \dfrac{1}{n}$ can be made smaller than any assumed difference, by increasing n,

   $\therefore$ to assume any difference leads to an absurdity.

7. $$\therefore \frac{B}{A} = \frac{D}{C}, \text{ whence } \frac{A}{B} = \frac{C}{D}.$$

466. Definition. The solid bounded by a lune and two semi-circles is called a **spherical wedge.**

The angle of the lune is called the *angle of the wedge.*
The word *ungula* is sometimes used for spherical wedge.

Corollaries. 1. *A lune is to the spherical surface on which it lies as the angle of the lune is to a perigon.*

For the spherical surface may be considered as a lune whose angle is a perigon.

2. *A spherical wedge is to the sphere of which it is a part as the angle of the wedge is to a perigon.*

In the proof of prop. X, if we should substitute the word *wedge* for the word *lune*, and consider the sphere as a wedge whose angle is a perigon, the corollary would evidently be proved.

Exercises. 703. To draw a plane tangent to a given sphere, from a point on the sphere. (See III, prop. XXVI.) Also, to draw one from an external point.

704. To find the locus of centers of spheres whose surfaces (1) pass through two given points; (2) are tangent to two given coplanar lines; (3) are tangent to two given planes. (As special cases, the lines may be parallel and the planes may be parallel.)

705. What is the locus of the centers of spheres whose surfaces (1) pass through the vertices of a given triangle? (2) are tangent to the sides of a given triangle?

706. To find the center of a sphere whose surface includes both a given circumference and a point not in the plane of that circumference. (As a special case, suppose the point is on the perpendicular to the plane of the given circle through the center.)

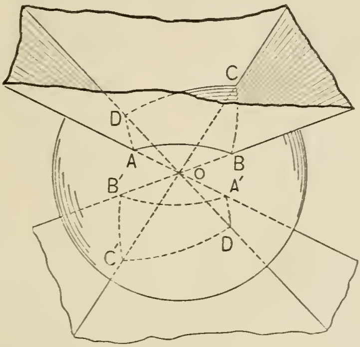
707. In the figure of prop. IX show that E, G, F, P are concyclic. Hence show that six circumferences intersect by threes in the circumcenters of the faces of a tetrahedron, and all intersect in the center of the circumscribed sphere.

708. To construct a sphere of given radius whose surface shall contain three given points.

709. Also, of given radius whose surface shall contain two given points and be tangent to a given plane.

THE RELATION OF SPHERICAL POLYGONS TO POLYHEDRAL ANGLES.

467. If the center of a sphere is at the vertex of a pyramidal space, the pyramidal surface cuts from the spherical surface two spherical polygons.



In the above figure the two polygons are $ABCD$, $A'B'C'D'$.

These polygons have their like-lettered angles and sides equal respectively.

For example. $\angle A = \angle A'$, since they have the same numerical measure as the opposite dihedral angles of planes $ADOD'A'$ and $ABOB'A'$. Also, $\widehat{AB} = \widehat{A'B'}$, since the central angles BOA and $B'OA'$ are equal.

468. But the equal elements of these polygons are arranged in reverse order. And as the polyhedral angles are called opposite and are proved (VI. prop. XXVI) symmetric, so the spherical polygons are called **opposite spherical polygons.** And since these have just been shown to have their corresponding elements equal but arranged in reverse order, they are called **symmetric spherical polygons.**

Thus all opposite polygons are symmetric; but since polygons can slide around on the sphere, it follows that symmetric polygons are not necessarily opposite, although they are congruent to opposite polygons.

469. Since the dihedral angles of the polyhedral angles have the same numerical measure as the angles of the spherical polygons, and the face angles of the former have the same numerical measure as the sides of the latter, it is evident that to each property of a polyhedral angle corresponds a reciprocal property of a spherical polygon, and *vice versa.* This relation appears by making the following substitutions:

Polyhedral Angles.	Spherical Polygons.
a. Vertex.	*a.* Center of Sphere.
b. Edges.	*b.* Vertices of Polygon.
c. Dihedral Angles.	*c.* Angles of Polygon.
d. Face Angles.	*d.* Sides.

470. In addition to the correspondences between polyhedral angles and spherical polygons, it will be observed that a relation exists between a straight line in a plane and a great-circle arc on a sphere. Thus, to a plane triangle corresponds a spherical triangle, to a straight line perpendicular to a straight line corresponds a great-circle arc perpendicular to a great-circle arc, etc. The word **arc** is always understood to mean **great-circle arc,** in the geometry of the sphere, unless the contrary is stated.

It is very desirable that every school have a spherical blackboard, with large wooden compasses for the drawing of both great and small circles. It is only by the use of such helps that students come to a clear knowledge of spherical geometry. If such a blackboard is at hand, it is recommended that many problems and exercises of Book I be investigated on the sphere. *E.g.* the problem, To bisect a given arc, corresponds to I, prop. XXXI, and the solutions are quite similar. Likewise the problems, To bisect a given angle, To draw a perpendicular to a given line from a given internal point, etc., have their corresponding problems in spherical geometry.

Exercise. 710. State without proof the proposition in the geometry of the sphere corresponding to the following: Every face angle of a convex polyhedral angle is less than a straight angle.

PROPOSITION XI.

471. Theorem.

(a) *In any trihedral angle, each face angle being less than a straight angle, the sum of any two face angles is greater, and their difference less, than the third angle.*

(a′) *In any spherical triangle, each side being less than a semicircumference, the sum of any two sides is greater, and their difference less, than the third side.*

Proof. In VI, prop. XXVII, with its cor. 1, (a) has been proved. Hence (a′) is also proved. § 469

PROPOSITION XII.

472. Theorem.

(a) *In any polyhedral angle, each face angle being less than a straight angle, any face angle is less than the sum of the remaining face angles.*

(a′) *In any spherical polygon, each side being less than a semicircumference, any side is less than the sum of the remaining sides.*

Proof. In VI, prop. XXVII, cor. 2, (a) has been proved. Hence (a′) is also proved. § 469

PROPOSITION XIII.

473. Theorem.

(a) *In any convex polyhedral angle the sum of the face angles is less than a perigon.*

(a′) *In any convex spherical polygon the sum of the sides is less than a circumference.*

Proof. In VI, prop. XXVIII, (a) has been proved. Hence (a′) is also proved. § 469

Proposition XIV.

474. Theorem.

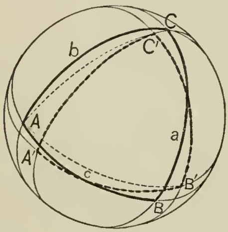
(a) *No face angle of a convex polyhedral angle is greater than a straight angle.*

(a′) *No side of a convex spherical polygon is greater than a semicircumference.*

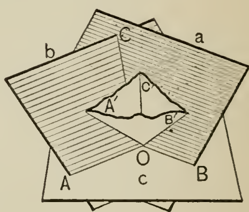
Proof. From § 461, (a′) is true.

Hence (a) is also proved. § 469

475. Definitions. If ABC is a *spherical triangle*, and A', B', C' are the *poles* of a, b, c, respectively, and if A and A', B and B', C and C' lie on the same side of a, b, c, respectively, then $\triangle A'B'C'$ is called the **polar triangle** of ABC.



If O-ABC is a *trihedral angle*, and OA', OB', OC' are *perpendiculars* to a, b, c, the faces opposite A, B, C, respectively, and if A and A', B and B', C and C' lie on the same side of a, b, c, respectively, then trihedral $\angle$ O-$A'B'C'$ is called the **polar trihedral angle** of O-ABC.



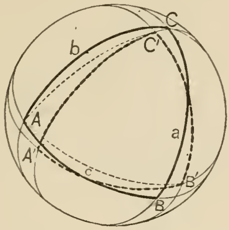
In referring to polar triangles ABC, $A'B'C'$, the above arrangement of elements will always be intended. Also, in referring to *symmetric* spherical triangles, ABC and $A'B'C'$, it will always be understood that $\angle A = \angle A'$, etc., and $\widehat{AB} = \widehat{A'B'}$, etc.

The polar triangle of ABC is often called *the polar* of ABC.

It is evident from the one-to-one correspondence of § 475, that to every proposition concerning polar triangles corresponds a proposition concerning polar trihedral angles, and *vice versa*.

PROPOSITION XV.

476. Theorem. *If one spherical triangle is the polar of a second, then the second is also the polar of the first.*



Given a spherical triangle, ABC, and $A'B'C'$ its polar.

To prove that $\triangle ABC$ is the polar of $\triangle A'B'C'$.

Proof. 1. In the figure suppose $\widehat{AC'}$, $\widehat{AB'}$, drawn.

2. Then, $\because B'$ is a pole of b,
$\therefore \widehat{AB'}$ is a quadrant. Prop. VI, cor. 2
Similarly, $\because C'$ is a pole of c,
$\therefore \widehat{AC'}$ is a quadrant.

3. $\therefore A$ is a pole of a'. Prop. VII
Similarly, B and C are poles of b' and c', respectively.

4. And $\because A$, A' are on the same side of a', and so for the other vertices and sides,
$\therefore \triangle ABC$ is the polar of $\triangle A'B'C'$. § 475

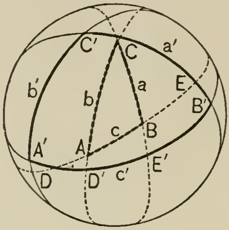
COROLLARY. *If one trihedral angle is the polar of a second, then the second is also the polar of the first.*

For from the one-to-one correspondence of § 475, the proof is evidently identical with the above.

NOTE. One triangle may fall entirely within or entirely without its polar; or one may be partly within and partly without the other. Similarly, one trihedral angle may fall entirely within or entirely without its polar trihedral angle, or may be partly within and partly without the latter.

Proposition XVI.

477. **Theorem.** *Any angle of a spherical triangle has the same numerical measure as the supplement of the opposite side of its polar.*



Given ABC, a spherical triangle, and $A'B'C'$ its polar.

To prove that the numerical measure of any angle C is $180° - c'$; of C', $180° - c$.

Proof. 1. Suppose a, b to cut c' in E', D', respectively, and a', b' to cut c in E, D, respectively.

2. Measure of $\angle C =$ that of $\widehat{D'E'}$. § 458, 2

But $\widehat{D'E'} = \widehat{A'E'} + \widehat{D'B'} - \widehat{A'B'}$

$= 90° + 90° - \widehat{A'B'} = 180° - c'$. Why 90°?

3. Similarly for $\angle C'$, substituting A, B, D, E, for A', B', D', E', and *vice versa*, in the above proof.

Corollaries. 1. *If two spherical triangles are mutually equiangular, their polars are mutually equilateral; if mutually equilateral, their polars are mutually equiangular.*

2. *The sum of the* angles *of a* spherical triangle *is greater than one and less than three straight angles.*

2'. *The sum of the* dihedral angles *of a* trihedral angle *is greater than one and less than three straight angles.*

For by prop. XIII (a′), $0 < a' + b' + c' < 360°$.

$\therefore$ by subtracting from $3 \cdot 180°$,

$$3 \cdot 180° > (180° - a') + (180° - b') + (180° - c') > 180°.$$

$\therefore$ by prop. XVI, $\quad 3 \cdot 180° > \angle A + \angle B + \angle C > 180°.$

478. Definitions. If $ABCD \ldots\ldots X$ is a *spherical polygon*, and A', B', C', D', ... are the *poles* of $\widehat{XA}$, $\widehat{AB}$, $\widehat{BC}$, $\widehat{CD}$, ..., respectively, and if A', B', ... lie on the same side of $\widehat{XA}$, $\widehat{AB}$, ... that the polygon does, then $A'B'C'D'$ ..... is called the **polar polygon** of $ABCD$ .....

If $O\text{-}ABCD \ldots\ldots X$ is a *polyhedral* $\angle$, and OA', OB', OC', OD', ... are ⊥s to planes OXA, OAB, OBC, ..., respectively, and if A', B', ... lie on the same side of planes OXA, OAB, OBC, ... that the polyhedral angle does, then $O\text{-}A'B'C'D'$ ..... is called the **polar polyhedral angle** of $O\text{-}ABCD$ .....

Polar trihedral angles are also called *supplemental trihedral angles.*

479. A spherical triangle is said to be **birectangular** if it has two right angles, **trirectangular** if it has three.

PROPOSITION XVII.

480. Theorem.

(a) *Two opposite or two symmetric trihedral angles are congruent if each has two equal dihedral angles, or two equal face angles.*

(a′) *Two opposite or two symmetric spherical triangles are congruent if each has two equal angles or two equal sides.*

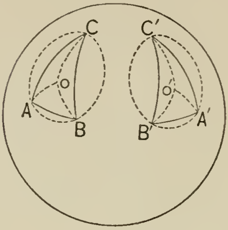
Proof for (a′). 1. Their sides and angles are respectively equal but arranged in reverse order. § 468

2. But if to the order ABC corresponds $B'A'C'$, and if $B' = A'$, then B' and A' may be interchanged.

3. Then to the order ABC will correspond $A'B'C'$, and the △s are congruent by superposition.

Proposition XVIII.

481. Theorem. *Two symmetric spherical triangles on the same sphere or on equal spheres are equal.*



Given two symmetric spherical triangles, ABC, $A'B'C'$, on the same sphere.

To prove that $\triangle ABC = \triangle A'B'C'$.

Proof. 1. The plane of A, B, C determines a small circle.

2. Let O be the pole of the $\odot$, and similarly for O' and spherical $\triangle A'B'C'$.

3. Then $\because$ side $\widehat{AB} =$ side $\widehat{A'B'}$, $\therefore$ chord $AB =$ chord $A'B'$. (In the figure they are not drawn because $\widehat{AB}$ is so nearly straight.) III, prop. III
Similarly for chords BC, $B'C'$, and CA, $C'A'$.

4. $\therefore$ plane $\triangle ABC \cong$ plane $\triangle A'B'C'$. I, prop. XII

5. $\therefore \odot ABC = \odot A'B'C'$, being circumscribed about congruent plane $\triangle$s. Why?

6. $\therefore \widehat{OA} = \widehat{OB} = \widehat{OC} = \widehat{O'A'} = \widehat{O'B'} = \widehat{O'C'}$. Why?

7. $\therefore$ spherical $\triangle$s $AOB \cong A'O'B'$, $BOC \cong B'O'C'$, $COA \cong C'O'A'$. §§ 468, 480 (a′)

8. $\therefore \triangle ABC = \triangle A'B'C'$. Ax. 2

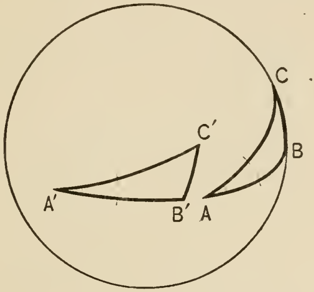
PROPOSITION XIX.

482. Theorem. *Two triangles on the same sphere, or on equal spheres, are either congruent or symmetric and equal if*

(a) *two* **sides** *and the included* **angle**

(b) *two* **angles** *and the included* **side**

of the one figure are equal to the corresponding parts of the other.



Proof. If the parts are arranged in the same order, the triangles can be brought into coincidence, as in I, props. I, II.

If they are arranged in reverse order, then one triangle is congruent to the triangle symmetric to the other. Why?

COROLLARY. *Two trihedral angles are either congruent or symmetric and equal if*

(a) *two* **face angles** *and the included* **dihedral angle**

(b) *two* **dihedral angles** *and the included* **face angle**

of the one figure are equal to the corresponding parts of the other.

For from the one-to-one correspondence of § 475, the proof is evidently identical with the above, without the labor of drawing the figures.

Proposition XX.

483. Theorem.

(a) *If a* **trihedral angle** *has two* **dihedral angles** *equal to each other, the opposite* **face angles** *are equal.*

(a′) *If a* **spherical triangle** *has two* **angles** *equal to each other, the opposite* **sides** *are equal.*

Given the $\triangle ABC$, with $\angle A = \angle B$.

To prove that $a = b$.

Proof. 1. Let $\triangle A'B'C'$ be symmetric to $\triangle ABC$, so that $a = a'$, $b = b'$, etc.

2. Then $\because \angle A = \angle B$,

$\therefore \angle A'$ must equal $\angle B'$, and the $\triangle$s are congruent and $a' = b$. Prop. XIX

3. But $\because a = a'$, and $a' = b$,

$\therefore a = b$, which proves (a′). Ax. 1

Hence (a) is also proved. § 469

Corollaries.

1. (a) *If a* **trihedral angle** *has its three dihedral angles equal, it has also its three face angles equal.*

(a′) *An equiangular* **spherical triangle** *is equilateral.*

In $\triangle ABC$, if $\angle A = \angle B$, then $a = b$; and if $\angle C$ also equals $\angle B$, c also equals b.

$\therefore$ if $\angle A = \angle B = \angle C$, $a = b = c$. Ax. 1

2. (a) *If a* **trihedral angle** *has two* **face angles** *equal to each other, the opposite* **dihedral angles** *are equal.*

(a′) *If a* **spherical triangle** *has two* **sides** *equal to each other, the opposite* **angles** *are equal.*

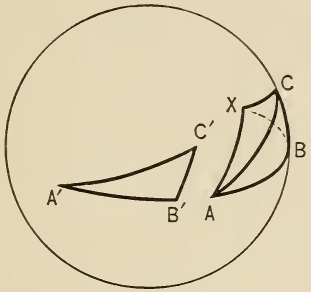
The proof is almost identical with that of I, prop. III, and hence is left for the student.

PROPOSITION XXI.

484. Theorem. *Two triangles on the same sphere, or on equal spheres, are either congruent or symmetric and equal if*

(a) *the three* sides (b) *the three* angles

of the one figure are equal to the corresponding parts of the other.



(a) Given $\triangle\triangle\ ABC, A'B'C'$, mutually equilateral, the sides being arranged in the same order; also $\triangle\ ACX$ symmetric to $\triangle\ A'B'C'$.

To prove that $\triangle\ ABC \cong \triangle\ A'B'C'$, $\triangle\ ABC$ is symmetric to $\triangle\ ACX$.

Proof. 1. Place $\triangle\ ACX$ as in the figure; draw $\overset{\frown}{BX}$.

Then $\angle BXC = \angle CBX$,

and $\angle AXB = \angle XBA$. Prop. XX, cor. 2

2. $\therefore \angle AXC = \angle CBA$,

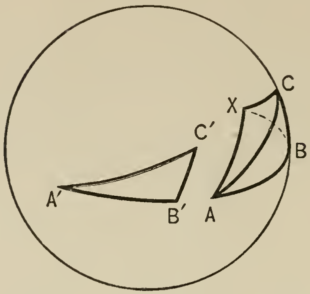
i.e. $\angle B = \angle X = \angle B'$. Ax. 3

Similarly with the other angles.

3. $\therefore \triangle\ A'B'C' \cong \triangle\ ABC$. Why?

4. And $\triangle\ ABC$ is symmetric to $\triangle\ ACX$. Why?

(b) Given (Let the student state it.)



To prove (Let the student state it.)

Proof. 1. Their polars are mutually equilateral. Why?

2. $\therefore$ their polars are congruent or symmetric. Why?

3. $\therefore \triangle ABC$ and $\triangle A'B'C'$ are mutually equilateral. Props. XV, XVI, cor. 1

4. $\therefore \triangle ABC \cong$ or symmetric to $\triangle A'B'C'$. Prop. XXI (a)

Corollary. *Two trihedral angles are either congruent or symmetric and equal if*

(a) *the three* **face angles** (b) *the three* **dihedral angles**

of the one figure are equal to the corresponding parts of the other.

Exercises. 711. A plane isosceles triangle can have its equal sides of any length. Discuss as to a spherical isosceles triangle on a given sphere.

712. As with plane triangles, the pole (circumcenter) may fall outside the triangle, or on a side. Prove theorem 481 for those cases.

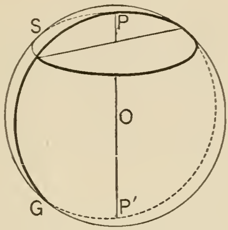
713. Prove I, prop. XII (a corresponding theorem of Plane Geometry) by the method of prop. XVIII.

714. Draw the figure of a spherical quadrilateral and its polar; also of a four-faced polyhedral angle and its polar.

715. Prove that if one spherical polygon is the polar of another, then the second is the polar of the first. State the reciprocal theorem for polyhedral angles. (The special case of the quadrilateral may be taken.)

PROPOSITION XXII.

485. Theorem. *For a great circle to be perpendicular to a small circle, it is necessary and it is sufficient that the circumference of the former pass through a pole of the latter.*



Given a small circle S, with P and P' its poles, G a great circle, and O the center of the sphere.

To prove that for G to be $\perp$ to S it is necessary and it is sufficient that its circumference pass through P.

Proof. 1. $PP' \perp S$. Def. pole

2. And if G passes through P it passes through PP', and $G \perp S$. VI, prop. XVIII

3. $\therefore$ it is *sufficient* that G contain P.

4. Furthermore it is *necessary;* for if $G \perp S$, then PP' lies in G or else $PP' \parallel G$. Why?

5. But PP' is not $\parallel$ to G, for each contains O. Why?

6. $\therefore$ it is *necessary*, and it is *sufficient*, that G contain P.

COROLLARIES. 1. *Through a point* X, *within or on a sphere, it is possible to pass one great circle perpendicular to a given circle* S, *and only one unless* X *is on the straight line through the poles of* S.

For PP' passes through the center O, and PP' and X determine a great circle $\perp$ S, unless X is on PP'. (Why?) May X be without the sphere?

2. *If the circumferences of two great circles are drawn perpendicular to a third circumference, they will intersect at the poles of the circle of that third circumference.*

486. Definition. If a great circle is perpendicular to a small circle, their **circumferences** are said to be **perpendicular** to each other.

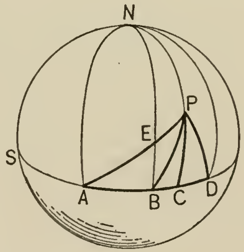
Proposition XXIII.

487. Theorem. *If from a point on a sphere arcs of great circles both perpendicular and oblique, are drawn to any circumference, then,*

1. The shorter perpendicular is less than any oblique;

2. Two obliques cutting off equal arcs from the foot of this perpendicular are equal;

3. Of two obliques cutting off unequal arcs from the foot of this perpendicular, the one cutting off the greater arc is the greater.



Given S, any circle of a sphere; P any point on the spherical surface; minor arcs, $\overset{\frown}{PC} \perp$ circumference S, $\overset{\frown}{PA}$, $\overset{\frown}{PB}$, $\overset{\frown}{PD}$ obliques; $\overset{\frown}{BC} = \overset{\frown}{CD}$, and $\overset{\frown}{AC} > \overset{\frown}{CD}$ or its equal $\overset{\frown}{BC}$.

To prove that (1) $\overset{\frown}{PC} < \overset{\frown}{PB}$;

(2) $\overset{\frown}{PB} = \overset{\frown}{PD}$;

(3) $\overset{\frown}{PA} > \overset{\frown}{PD}$ or its equal $\overset{\frown}{PB}$.

Proof. 1. Suppose N the pole of S, on the same side of S as P; draw $\widehat{NA}$, $\widehat{NB}$, $\widehat{ND}$.

2. $\widehat{CP}$ produced passes through N. Prop. XXII, cor. 2

3. $\widehat{NB}$, or its equal $\widehat{NC}$, $< \widehat{NP} + \widehat{PB}$. Prop. XI, (a′)

4. $\therefore \widehat{PC} < \widehat{PB}$. Why?

5. The radius of $\odot S$ through C is $\perp$ to chord BD and bisects it as at Q (not shown in figure). Why?

6. $\therefore DB \perp$ plane NPC. Why?

7. $\therefore$ plane $\measuredangle\, DQP = PQB$, Why?

and plane $\triangle\, QPD \cong QPB$. I, prop. I

8. $\therefore \widehat{PB} = \widehat{PD}$. III, prop. IV

9. $\widehat{PE} + \widehat{EB} > \widehat{PB}$, and $\widehat{EA} > \widehat{EB}$. Why?

10. $\therefore \widehat{PE} + \widehat{EA}$, or $\widehat{PA}$, $> \widehat{PB}$.

488. Definition. The excess of the sum of the angles of a spherical n-gon over $(n-2)$ straight angles is called the **spherical excess** of the n-gon.

Hence the spherical excess of a 2-gon (lune), 3-gon (triangle), 4-gon, ... is the excess of the sum of its angles over 0, 1, 2, ... straight angles.

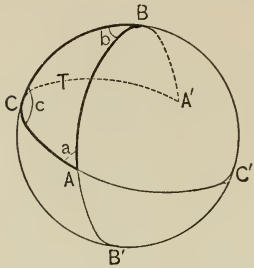
Exercises. 716. Prove that if in prop. XVI the word *polygon* is substituted for *triangle*, the resulting theorem is true, and state the corollary that follows from it, analogous to corollary 1 of prop. XVI.

717. What is meant by the spherical excess of a spherical decagon? What is the spherical excess, in degrees, of a triangle whose angles are 75°, 90°, 100°?

718. What is the spherical excess, in radians, of a triangle whose angles are 80°, 90°, 100°? Also of a triangle whose angles are 1, 2, and 3 radians, respectively?

Proposition XXIV.

489. Theorem. *A spherical triangle equals a lune whose angle is half the spherical excess of the triangle.*



Given T, a spherical triangle, with angles a, b, c.

To prove that $T =$ a lune whose angle is $\frac{1}{2}(a + b + c - \text{st.} \angle)$.

Proof. 1. Let $A, B, C =$ lunes of $\angle$s a, b, c, respectively (in the figure they are AA', BB', CC'), and $S =$ surface of sphere.

2. $\triangle$s $AB'C'$ and $A'BC$ are mutually equilateral, for $\overset{\frown}{AC'} + \overset{\frown}{CA} =$ semicircumference $= \overset{\frown}{A'C} + \overset{\frown}{CA}$; hence $\overset{\frown}{A'C} = \overset{\frown}{AC'}$, and so for the other sides. Ax. 3

3. $\therefore \triangle AB'C' = \triangle A'BC$, so that $T + \triangle AB'C' =$ lune A. Prop. XXI

4. $\therefore A + (B - T) + (C - T) = \frac{1}{2} S$, Ax. 8

or $T = \frac{1}{2}(A + B + C - \frac{1}{2} S)$. Axs. 3, 7

5. But $\because \frac{1}{2} S =$ a lune whose $\angle$ is a st. $\angle$, § 462

$\therefore T =$ a lune whose $\angle$ is $\frac{1}{2}(a + b + c - \text{st.} \angle)$.

Corollary. *A spherical polygon equals a lune whose angle is half the spherical excess of the polygon.*

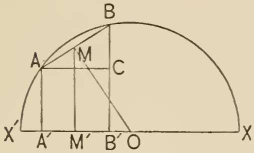
For the polygon can be cut into triangles as in Plane Geometry.

The practical method of measuring a spherical polygon is given in § 493, cor. 3.

4. THE MENSURATION OF THE SPHERE.

PROPOSITION XXV.

490. Theorem. *The area of the surface of a sphere of radius* r *is* $4\pi r^2$.



Proof. 1. A semicircle cut off by a diameter $X'X$, revolving about $X'X$ as an axis, generates a sphere.

2. Let AB be one of a number of chords inscribed in arc XBX', forming half of a regular polygon.
Let $OM \perp AB$, thus bisecting it; III, prop. V
let AA', MM', BB', all be $\perp$ to $X'X$, and $AC \parallel X'X$.

3. Then AB, revolving about axis $X'X$, generates the surface $l = 2\pi \cdot AB \cdot M'M$. Prop. III, cor. 2

4. But $\because \triangle ACB \backsim \triangle MM'O$, Why?
$\therefore OM : M'M = AB : AC = AB : A'B'$.

5. $\therefore AB \cdot M'M = A'B' \cdot OM$. IV, prop. I

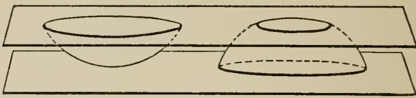
6. $\therefore l = 2\pi \cdot A'B' \cdot OM$. Subst. in 3

7. Summing for all frustums, including two cones, the sum of the lateral surfaces $= 2\pi \cdot OM \cdot (X'A' + A'B' + \cdots\cdots) = 2\pi \cdot OM \cdot 2r$. Axs. 2, 8

8. But if the number of sides increases indefinitely, the sum of the lateral surfaces $\doteq$ surface of sphere, s, and $OM \doteq r$;
$\therefore s = 2\pi \cdot r \cdot 2r = 4\pi r^2$. IV, prop. IX, cor. 1

491. Definitions. That part of a spherical surface which is included between two parallel planes which cut or touch the surface, is called a **zone.**

The solid bounded by the zone and the two parallel planes is called a **spherical segment.**



Zones and spherical segments. In first figure, lower base is zero.

The distance between the two parallel planes determining a zone and a spherical segment is called the **altitude** of the zone and the segment.

The circumferences in which the planes intersect the spherical surface are called the **bases of the zone,** and the circles are called the **bases of the segment.**

In case of tangent planes the bases may one or both reduce to zero. If one base only reduces to zero, the zone, or segment, is said to have *one base.*

492. Definition. As a plane angle is often said to be measured by the ratio of the intercepted arc to the whole circumference (§ 256), so **a polyhedral angle is said to be measured** by the ratio of the intercepted spherical polygon to the whole spherical surface.

The practical method of measuring a polyhedral angle is given in § 493, cor. 4.

493. Corollaries. 1. *The area of a zone of altitude* a, *on a sphere of radius* r, *is* $2\pi ra$.

For, prop. XXV, step 7, the sum of the lateral surfaces may approach as their limit a zone, in which case $X'A' + A'B' + \cdots \doteq a$, and $OM \doteq r$.

2. *The area of a lune of angle* α *(expressed in radian measure) on a sphere of radius* r, *is* $2\alpha r^2$.

By prop. X, $l : 4\pi r^2 = \alpha : 2\pi$.

3. *The area of a spherical polygon of spherical excess* α *(expressed in radian measure) is* αr^2.

For by prop. XXIV, the polygon equals the lune whose angle is $\alpha/2$.

$\therefore$ the area $= 2 \cdot \frac{\alpha}{2} \cdot r^2 = \alpha r^2$, by cor. 2.

4. *The measure of a polyhedral angle whose intercepted spherical polygon has a spherical excess α is* $\frac{\alpha}{4\pi}$.

For by definition, § 490, it is $\frac{\alpha r^2}{4\pi r^2}$.

5. *The area generated by a chord of a circle revolving about a diameter which does not cut it, equals 2π times the product of its projection on that diameter, and the distance from the center to the chord.* (Why?)

6. *The areas of two spheres are proportional to the squares of their radii.*

For $\frac{a}{a'} = \frac{4\pi r^2}{4\pi r'^2} = \frac{r^2}{r'^2}$.

Exercises. 719. What is the area of a spherical triangle the sum of whose angles is 4 radians, on a sphere of radius 1 ft.?

720. Also of one the sum of whose angles is 270°, r being 2 ft.?

721. Also of one the sum of whose angles is 180°, on any sphere?

722. Also of one the sum of whose angles is 237° 29′, r being 10 in.?

723. Also of a spherical quadrilateral the sum of whose angles is 417° 29′, on a sphere of radius 2 in.?

724. Also of a spherical pentagon the sum of whose angles is 4 straight angles, on a sphere of radius 5 in.?

725. What is the measure in radians of a polyhedral angle the spherical excess of whose intercepted spherical polygon is 8π?

726. What is the ratio of a trihedral angle the sum of the angles of whose intercepted spherical triangle is 1.5π radians, to a tetrahedral angle the sum of the angles of whose intercepted spherical quadrilateral is 2.5π radians?

727. What is the area of a spherical triangle whose angles are 70°, 80°, 90°, on a sphere of diameter 20 in.?

728. Show that a trirectangular triangle is its own polar.

729. The locus of points on a sphere, from which great-circle arcs perpendicular to the arms of an angle are equal, is the great-circle arc bisecting that angle.

Proposition XXVI.

494. Theorem. *Two solids lying between two parallel planes, and such that the two sections made by any plane parallel to the given planes are equal, are themselves equal.*

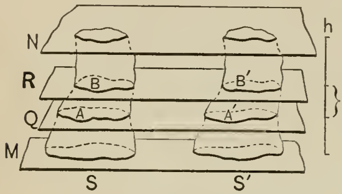


Fig. 1.

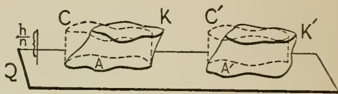


Fig. 2.

Given two solids, S, S', lying between parallel planes, M, N, and such that the two sections A, A', or B, B', ..., made by any plane Q, or R, ..., are equal, *i.e.* $A = A'$, $B = B'$, ...

To prove that $S = S'$.

Proof. 1. Let K, K' be two segments of S, S', lying between the sections A and B, and A' and B'; let the altitude of K, K' be $1/n$ of the altitude h of S and S'.

2. Suppose two straight lines to move so as always to be perpendicular to Q, and to touch the perimeters of A, A', thus generating two cylinders (or prisms, or combinations of cylinders and prisms) of altitude h/n as in Fig. 2. As the volumes of both prisms and cylinders are expressed by the same formula, $v = bh$, we may speak of these solids as cylinders, C, C'.

3. Then $C = C'$, since they have equal bases and altitudes; and so for other pairs of cylinders described in the same way, with altitude h/n.

4. $\therefore$ the sum of the solids like $C =$ the sum of the solids like C', whatever n equals.

5. But if n increases indefinitely, h/n decreases indefinitely, and it is evident that the sum of the solids like $C \doteq S$, while the sum of the solids like $C' \doteq S'$.

6. $\therefore S = S'$. IV, prop. IX, cor. 1

495. This important proposition is known as **Cavalieri's theorem.** It will be seen that VII, prop. XV, is merely a special case of this proposition. We shall base the mensuration of the volume of the sphere upon it. Solids of this kind are often called **Cavalieri bodies.**

Exercises. **730.** A spherical triangle is to the surface of the sphere as the spherical excess is to eight right angles.

731. The locus of points on a sphere, from which great-circle arcs to two fixed points on the sphere are equal, is the circumference of a great circle perpendicular to the arc joining those points at its mid-point.

732. There is evidently a proposition of plane geometry analogous to Cavalieri's theorem, beginning, "Two plane surfaces lying between two parallel lines ....." State this proposition and prove it.

733. From ex. 732 prove that triangles having equal bases and equal altitudes are equal.

734. What is the ratio of the surface of a sphere to the *entire* surface of its hemisphere?

735. Prove that the areas of zones on equal spheres are proportional to their altitudes.

736. Find the ratio of the surfaces of two spheres, in terms of their radii, r_1 and r_2.

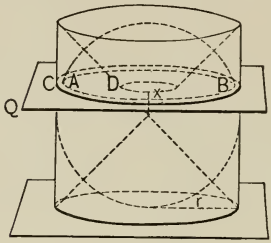
737. What is the ratio of the area of a great circle of a sphere to the area of its spherical surface?

738. If a meter is 0.0000001 of a quadrant of the earth's circumference, and the earth is assumed to be a sphere, how many square myriameters of surface has the earth?

739. What is the radius of the sphere whose area is 1 square unit? Answer to 0.001.

Proposition XXVII.

496. Theorem. *The volume of a sphere of radius* r *is expressed by the formula* $v = \frac{4}{3}\pi r^3$.



Proof. 1. Suppose the sphere circumscribed by a cylinder, and suppose two cones formed with the bases of the cylinder as their bases, and their vertices at the center of the sphere.

Suppose the solid to be cut by a plane Q, parallel to the bases, and x distant from the center of the sphere.

2. Then since x also equals the radius of the ⊙ cut from the cone, because the altitude of the cone equals the radius of its base,

$\therefore$ area of ring CD between cone and cylinder

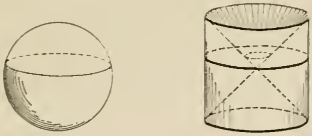
$$= \pi r^2 - \pi x^2,$$
$$= \pi (r^2 - x^2).$$

3. But the area of the ⊙ AB cut from the sphere is also $\pi (r^2 - x^2)$, because its radius is $\sqrt{r^2 - x^2}$.

4. $\therefore$ the sphere and the difference between the cone and cylinder are two Cavalieri bodies, and $\therefore$ they are equal. Prop. XXVI

5. $\therefore v = \pi r^2 \cdot 2r - \pi r^2 \cdot \frac{2r}{3} = \frac{4}{3}\pi r^3.$ Why?

COROLLARIES. 1. *The volume of a sphere equals two-thirds the volume of the circumscribed cylinder.* (Archimedes's theorem.)



The volume of the circumscribed cylinder is evidently $\pi r^2 \cdot 2r$, or $2\pi r^3$. And $\frac{4}{3}\pi r^3$ is $\frac{2}{3}$ of $2\pi r^3$.

2. *The volume of a sphere equals the product of its surface by one-third of its radius.*

For the surface is $4\pi r^2$, prop. XXV; and $\frac{4}{3}\pi r^3 = \frac{1}{3} r \cdot 4\pi r^2$.

3. *The volumes of two spheres are proportional to the cubes of their radii.*

4. *The volume of a spherical segment of one base, of altitude* a, *is expressed by the formula* $v = \frac{1}{3}\pi a^2(3r - a)$.

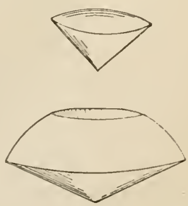
For, as in the theorem, it equals the difference between a circular cylinder of radius r and altitude a, and the frustum of a cone, of the same altitude and with bases of radii r and $(r - a)$.

$$\therefore v = \pi r^2 a - \tfrac{1}{3}\pi a[r^2 + (r-a)^2 + r(r-a)] \qquad \text{Prop. IV, cor. 1}$$
$$= \tfrac{1}{3}\pi a^2(3r - a).$$

497. Definitions. A **spherical sector** is the portion of a sphere generated by the revolution of a circular sector about any diameter of its circle as an axis.

The **base** of the spherical sector is the zone generated by the arc of the circular sector, and the **altitude** is the altitude of that zone.

If the base of the spherical sector is a zone of one base only, the spherical sector is called a **spherical cone**.



Spherical sectors. The upper one a spherical cone.

Corollaries. 1. *The volume of a spherical cone, whose base* b *has an altitude* a, *is expressed by the formula* $v = \frac{2}{3}\pi r^2 a$, *or* $v = \frac{1}{3}br$.

For it evidently equals the sum of a cone and a spherical segment of one base. What does the latter equal, by § 496, cor. 4? Show that the cone $= \frac{1}{3}\pi(r - a)[r^2 - (r - a)^2]$. Then add the results, and show that the sum is $\frac{2}{3}\pi r^2 a$. But $b = 2\pi ra$. (Why?)

2. *The volume of a spherical sector, whose base is* b *and altitude* a, *is expressed by the formula* $v = \frac{2}{3}\pi r^2 a$, *or* $v = \frac{1}{3}br$.

For it equals the difference between two spherical cones.

Suppose these to have altitudes a_1, a_2,
and bases b_1, b_2,
and volumes v_1, v_2, respectively.

Then $v = v_1 - v_2 = \frac{2}{3}\pi r^2 a_1 - \frac{2}{3}\pi r^2 a_2 = \frac{2}{3}\pi r^2(a_1 - a_2)$.

But $a_1 - a_2 = a$. (Why?)

$\therefore v = \frac{2}{3}\pi r^2 a$. Now show that $v = \frac{1}{3}br$.

Exercises. 740. Show that if the directrix of a cylinder is the circumference of a great circle of a sphere, and the generatrix is perpendicular to that circle, and the bases of the cylinder are circles tangent to the sphere, then the cylinder may be said to be circumscribed about the sphere.

741. After considering ex. 740, show that the surface of a sphere is two-thirds the entire surface of the circumscribed cylinder. (Archimedes.)

742. Find the ratio of a spherical surface to the cylindrical surface of the circumscribed cylinder.

743. What is the radius of that sphere the number of square units of whose area equals the number of linear units in the circumference of one of its great circles?

744. What is the ratio of the entire surface of a cylinder circumscribed about a sphere to the entire surface of its hemisphere?

745. What is the area of the entire surface of a spherical segment the radii of whose bases are r_1, r_2, the radius of the sphere being r?

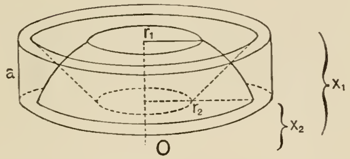
746. A cone has for its base a great circle of a sphere, and for its vertex a pole of that circle. Find the ratio of the curved surfaces of the cone and hemisphere; of the entire surfaces.

747. Show that the area of a zone of one base (the other base is zero) equals that of a circle whose radius is the chord of the generating arc.

PROPOSITION XXVIII.

498. Theorem. *The volume of a spherical segment of altitude* a, *whose bases have radii* r_1, r_2, *is expressed by the formula*

$$v = \tfrac{1}{6}\pi a\left[3(r_1^2 + r_2^2) + a^2\right], \text{ or}$$
$$v = \tfrac{1}{2}\pi a(r_1^2 + r_2^2) + \tfrac{1}{6}\pi a^3.$$



Proof. 1. Let the above figure represent a segment cut from the figure of prop. XXVII.

2. Then if the distances of the circles of radii r_1 and r_2, from the center O, are x_1 and x_2, respectively, the radii of the bases of the frustum of the cone are x_1 and x_2. Why?

3. $v =$ cylinder $-$ frustum, Prop. XXVII, step 4

$= \pi r^2 a - \frac{1}{3}\pi a(x_1^2 + x_2^2 + x_1 x_2)$ Prop. IV, cors. 1, 6

$= \frac{1}{6}\pi a(6r^2 - 2x_1^2 - 2x_1x_2 - 2x_2^2)$

$= \frac{1}{6}\pi a\left[3(r^2 - x_1^2) + 3(r^2 - x_2^2) + (x_1 - x_2)^2\right].$

4. But $\because a = x_1 - x_2$, and $r_1^2 = r^2 - x_1^2$,

and $r_2^2 = r^2 - x_2^2$,

$$\therefore v = \tfrac{1}{6}\pi a\left[3(r_1^2 + r_2^2) + a^2\right]$$
$$= \tfrac{1}{2}\pi a(r_1^2 + r_2^2) + \tfrac{1}{6}\pi a^3.$$

Exercise. 748. Within an equilateral triangle of side s is inscribed a circle; the triangle revolves about one of its axes of symmetry, thus generating a sphere and a cone. Find the ratio of their curved surfaces.

749. Also find the ratio of their entire surfaces.

5. SIMILAR SOLIDS.

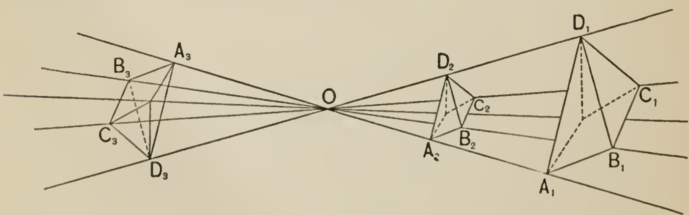
499. The definitions of **similar systems of points** and **similar figures** given in §§ 257, 258 are not limited to plane figures. The lines forming the pencil need not be coplanar. In case they are not coplanar, a pencil of lines is often called a **sheaf of lines.** The similar figures may then be plane, or they may be curved surfaces, or solids, etc. The definitions and corollaries on pages 182–184 are therefore the same for solid figures as for plane, and should be reviewed as part of this section.

Polyhedra which have equal face and equal dihedral angles, and equal edges, but have these parts arranged in reverse order, are said to be **symmetric.**

The polyhedral angles are then respectively symmetric.

Proposition XXIX.

500. Theorem. *If two polyhedra are similar, their corresponding face and dihedral angles are equal, their corresponding polyhedral angles are either congruent or symmetric, and their corresponding edges are in proportion, the constant ratio being the ratio of similitude.*



Given two similar polyhedra, $A_1B_1C_1$ ..... and $A_2B_2C_2$ ....., or $A_1B_1C_1$ ..... and $A_3B_3C_3$ .....

To prove that (1) $\angle B_1A_1D_1 = \angle B_2A_2D_2$ or $\angle B_3A_3D_3$;

(2) dihedral angle with edge $A_1B_1 =$ dihedral angle with edge A_2B_2, or with edge A_3B_3;

(3) polyhedral angle $A_1 \cong$ polyhedral angle A_2 and is symmetric to polyhedral angle A_3 as arranged in the figure; and

(4) $A_1B_1 : A_2B_2 =$ the ratio of similitude.

Proof. 1. Let the polyhedra be placed in perspective (§ 259), O the center of similitude, $A_1B_1C_1$ ..... and $A_2B_2C_2$ ..... on one side of O, and $A_3B_3C_3$ ..... on the other.

2. Then as in IV, prop. XX, $A_1B_1 \parallel A_2B_2 \parallel A_3B_3$ and $D_1A_1 \parallel D_2A_2 \parallel D_3A_3$.

3. $\therefore \angle B_1A_1D_1 = \angle B_2A_2D_2 = \angle B_3A_3D_3$, and similarly for other face angles, which proves (1). VI, prop. V

4. The trihedral $\angle A_1 \cong \angle A_2$ because the face $\measuredangle$s are respectively equal and similarly placed, and is symmetric to $\angle A_3$ because the face angles are respectively equal and placed in reverse order. Prop. XXI, cor.

5. So for the other trihedral $\measuredangle$s. And $\because$ polyhedral $\measuredangle$s, as D_1, D_2, D_3, can be cut into congruent or symmetric trihedral $\measuredangle$s similarly placed, as by the planes $A_1C_1D_1$, $A_2C_2D_2$, $A_3C_3D_3$, they too are congruent or symmetric.

6. $\therefore$ the dihedral $\measuredangle$s are equal, which proves (2), and the corresponding polyhedral $\measuredangle$s are congruent or symmetric, which proves (3).

7. The corresponding edges, as A_1B_1, A_2B_2, A_3B_3, being corresponding sides of similar $\triangle$s OA_1B_1, OA_2B_2, OA_3B_3, have the ratio of similitude, which proves (4).

COROLLARIES. 1. *If the ratio of similitude is 1, the polyhedra are either congruent or symmetric.*

2. *Corresponding faces of similar polyhedra are proportional to the squares of any two corresponding edges.*

Step 7, and V, prop. IV.

PROPOSITION XXX.

501. Theorem. *Two similar polyhedra can be divided into the same number of tetrahedra similar each to each and similarly placed.*

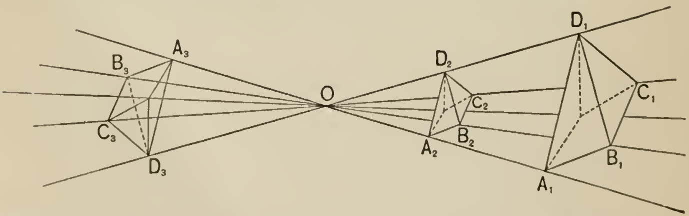
Proof. 1. In the figure below, the plane of A_1, C_1, D_1 and the plane of A_2, C_2, D_2 cut off tetrahedra $A_1B_1C_1D_1$, $A_2B_2C_2D_2$.

2. Any point P_1 in the one has a corresponding point P_2 in the other, such that $OP_1 : OP_2 =$ the ratio of similitude. Why?

3. Hence the tetrahedra are similar.

PROPOSITION XXXI.

502. Theorem. *The volumes of similar polyhedra are to each other as the cubes of their corresponding edges.*



Given the similar polyhedra $A_1B_1C_1$ ....., $A_2B_2C_2$ ....., $A_3B_3C_3$ ....., having volumes v_1, v_2, v_3, respectively.

To prove that $v_1 : v_2 = A_1B_1^3 : A_2B_2^3$, $v_1 : v_3 = A_1B_1^3 : A_3B_3^3$.

Proof. 1. Place the polyhedra in perspective, as in the figure, the center of similitude being O.

Divide the polyhedra into similar tetrahedra, similarly placed, $A_1B_1C_1D_1$, $A_2B_2C_2D_2$, $A_3B_3C_3D_3$, being corresponding tetrahedra. Prop. XXX

Let t_1, t_2, t_3 represent the volumes of these respective tetrahedra, p_1, p_2, p_3 their altitudes from D_1, D_2, D_3, and a_1, a_2, a_3 the areas of $\triangle A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$.

2. Then $\because t_1 = \frac{1}{3} p_1 a_1$,

and $t_2 = \frac{1}{3} p_2 a_2$,

$\therefore t_1 : t_2 = p_1 a_1 : p_2 a_2$.

3. But $a_1 : a_2 = A_1B_1^2 : A_2B_2^2$, V, prop. IV

and $p_1 : p_2 = D_1A_1 : D_2A_2 = A_1B_1 : A_2B_2$. IV, prop. XX

4. $\therefore p_1a_1 : p_2a_2 = A_1B_1^3 : A_2B_2^3$. IV, prop. VII, cor.

5. $\therefore t_1 : t_2 = A_1B_1^3 : A_2B_2^3$. From steps 2, 4

Similarly the other tetrahedra are proportional to the cubes of their corresponding edges, which edges are proportional to the particular edges A_1B_1 and A_2B_2.

6. $\therefore$ the sum of the tetrahedra making up the polyhedron $A_1B_1C_1$ ..... has the same ratio to the sum of the tetrahedra making up the polyhedron $A_2B_2C_2$ ..... as $A_1B_1^3$ has to $A_2B_2^3$, or

$$v_1 : v_2 = A_1B_1^3 : A_2B_2^3.$$

Similarly $v_1 : v_3 = A_1B_1^3 : A_3B_3^3$,

$$\begin{aligned} v_2 : v_3 &= A_2B_2^3 : A_3B_3^3, \\ &= B_2C_2^3 : B_3C_3^3, \\ &= C_2D_2^3 : C_3D_3^3, \text{ etc.} \end{aligned}$$

Proposition XXXII.

503. Theorem. *Any two spheres are similar.*

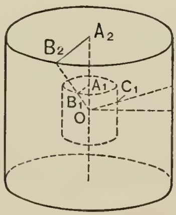
Proof. Let the spheres be placed in concentric position.

Then ∵ the ratio of their radii is constant any point on the surface of the one has on the surface of the other its similar point, with respect to the center, the ratio being $r_1 : r_2$.

∴ the spheres are similar.

Proposition XXXIII.

504. Theorem. *Two right circular cylinders are similar if their elements have the same ratio as the radii of their bases.*



Proof. 1. Let the cylinders have the radii r_1, r_2, and the altitudes h_1, h_2, respectively, and be placed with their axes in the same line, their mid-points coinciding at O.

Let the semi-altitudes be OA_1, OA_2, and let a line from O cut the bases in B_1, B_2, not necessarily on the circumferences, and one from O cut the cylindrical surfaces in C_1, C_2, respectively.

2. Then ∵ the altitudes are proportional to the radii,

$$\therefore OA_1 : OA_2 = r_1 : r_2.$$

And $\because A_1B_1 \parallel A_2B_2$, Why?

$\therefore OB_1 : OB_2 = OA_1 : OA_2 = r_1 : r_2.$ IV, prop. X

3. And $\because$ the axes coincide $\therefore$ the elements are parallel, and $\therefore OC_1 : OC_2 = r_1 : r_2$.

$\therefore$ the points of the respective cylinders are similar with respect to O as a center.

COROLLARIES. 1. *The areas of the cylindrical surfaces of two similar cylinders are proportional to the squares of their altitudes.*

For $a_1 = 2\,\pi r_1 h_1$ and $a_2 = 2\,\pi r_2 h_2$.

$$\therefore \frac{a_1}{a_2} = \frac{r_1 h_1}{r_2 h_2}.$$

But $\because \frac{r_1}{r_2} = \frac{h_1}{h_2}$ by prop. XXXIII,

$$\therefore \frac{a_1}{a_2} = \frac{h_1^2}{h_2^2}.$$

2. *The volumes of two similar right circular cylinders are proportional to the cubes of their altitudes.*

PROPOSITION XXXIV.

505. Theorem. *Two right circular cones are similar if their altitudes have the same ratio as the radii of their bases.*

Place the bases in concentric position. The proof is then so similar to that of prop. XXXIII that it is left for the student.

COROLLARIES. 1. *The areas of the surfaces of two similar right circular cones are proportional to the squares of their altitudes.*

2. *The volumes of two similar right circular cones are proportional to the cubes of their altitudes.*

EXERCISES.

750. The mean radii of the earth and moon are respectively 3956 miles, 1080.3 miles. Show that their volumes are as 49 to 1, nearly.

751. The mean diameter of the planet Jupiter being 86,657 miles, find the ratio of its volume to that of the earth.

752. The sun's diameter is about 109 times the earth's. Find the ratio of their volumes.

753. What is the radius of that sphere whose number of square units of surface equals the number of cubic units of volume?

754. Also of that whose number of cubic units of volume equals the number of square units of area of one of its great circles.

755. Also of that whose number of cubic units of volume equals the number of linear units of the circumference of a great circle.

756. Two planes cut a sphere of radius 1 m, at distances 0.8 m and 0.5 m from the center. Find (1) the area of the zone between them, (2) the volume of the corresponding spherical segment.

757. A solid cylinder 20 cm long and 2 cm in diameter is terminated by two hemispheres. The solid is melted and molded into a sphere. Find the diameter of the sphere.

758. A meter was originally intended to be 0.0000001 of a quadrant of the circumference of the earth. Assuming it to be such, and the earth to be a sphere, find its radius in kilometers.

759. A cone, a sphere, and a cylinder have the same altitudes and diameters. Show that their volumes are in arithmetical progression.

760. Given a sphere of radius 10. How far from its center must the eye be in order to see one-fourth of its surface?

761. If a tetrahedron is cut by a plane parallel to one of its faces, the tetrahedron cut off is similar to the first.

762. The areas of the surfaces of two similar polyhedra are proportional to the squares of their corresponding edges.

506. NUMERICAL TABLES.

FORMULÆ OF MENSURATION. The numbers refer to the pages. Abbreviations: b, base; h, altitude; r, radius; a, area; c, circumference; p, perimeter; s, slant height; v, volume; m, mid-section.

Parallelogram, 202, $a = bh$. Circle, 217, 224, $c = 2\pi r$.
Triangle, 202, $a = \frac{1}{2}bh$. $a = \pi r^2$.
Trapezoid, 202, $a = \frac{1}{2}(b + b')h$. Arc, 223, $= \alpha \cdot r$.

Parallelepiped, 307, $v = bh$.
Prism, 307, $v = bh$.
Lateral area, right prism, 298, $a = ph$.
Prismatoid, 314, $v = \frac{1}{6}h(b + b' + 4m)$.
Pyramid, 313, $v = \frac{1}{3}bh$.
Lateral area, regular pyramid, 309, $a = \frac{1}{2}ps$.
Frustum of pyramid, 315, $v = \frac{1}{3}h(b + b' + \sqrt{bb'})$.
Lateral area, frustum of regular pyramid, 309, $a = \frac{1}{2}(p + p')s$.

Right circular cylinder, 324, 325, $v = bh = \pi r^2 h$. Lateral $a = ch = 2\pi rh$.
Right circular cone, 324, 325, $v = \frac{1}{3}bh = \frac{1}{3}\pi r^2 h$.
Lateral $a = \frac{1}{2}cs = \pi rs$.
Frust. of rt. circ. cone, 325, $v = \frac{1}{3}\pi h(r_1^2 + r_2^2 + r_1 r_2)$.

Sphere, 355, 360, $v = \frac{4}{3}\pi r^3$. $a = 4\pi r^2$.
Lune, 356, $a = 2\alpha r^2$.
Spherical polygon, 356, $a = \alpha r^2$.
Zone, 356, $a = 2\pi rh$.
Spherical segment, 363, $v = \frac{1}{6}\pi h[3(r_1^2 + r_2^2) + h^2]$.
Spherical sector, 362, $v = \frac{2}{3}\pi r^2 h = \frac{1}{3}br$.

MOST IMPORTANT EXPRESSIONS INVOLVING π.

$\pi = 3.141593.$	$1/\pi = 0.31830989.$	$180°/\pi = 57°.29578.$
$\pi/4 = 0.785398.$	$\pi^2 = 9.86960440.$	$\pi/180 = 0.01745.$
$\pi/3 = 1.047198.$	$\sqrt{\pi} = 1.77245385.$	Approximate values;
$\frac{4}{3}\pi = 4.188790.$	$1/\sqrt{\pi} = 0.56418958.$	$\pi = \frac{22}{7} = 3\frac{1}{7}, \frac{355}{113}.$

CERTAIN NUMERICAL RESULTS FREQUENTLY USED.

$\sqrt{2} = 1.4142.$	$\sqrt{10} = 3.1623.$	$\sqrt[3]{5} = 1.7100.$
$\sqrt{3} = 1.7321.$	$\sqrt{\frac{1}{2}} = 0.7071.$	$\sqrt[3]{6} = 1.8171.$
$\sqrt{5} = 2.2361.$	$\sqrt[3]{2} = 1.2599.$	$\sqrt[3]{7} = 1.9129.$
$\sqrt{6} = 2.4495.$	$\sqrt[3]{3} = 1.4422.$	$\sqrt[3]{9} = 2.0801.$
$\sqrt{7} = 2.6458.$	$\sqrt[3]{4} = 1.5874.$	$\sqrt[3]{10} = 2.1544.$

507. BIOGRAPHICAL TABLE.

The following table includes only those names mentioned in this work, although numerous others might profitably be considered by the student. The history of geometry may be said to begin in Egypt, the work of Ahmes, copied from a treatise of about 2500 B.C., containing numerous geometric formulæ. The scientific study of the subject did not begin, however, until Thales visited that country, and carried the learning of the Egyptians back to Greece. The period of about four hundred years from Thales to Archimedes may be called the golden age of geometry. The contributions of the latter to the mensuration of the circle and of certain solids practically closed the scientific study of the subject in ancient times. Only a few contributors, such as Hero, Ptolemy, and Menelaus, added anything of importance during the eighteen hundred years which preceded the opening of the seventeenth century. Within the past three hundred years several important propositions and numerous improvements in method have been added, but the great body of elementary plane geometry is quite as Euclid left it. In recent times a new department has been created, known as Modern Geometry, involving an extensive study of loci, collinearity, concurrence, and other subjects beyond the present range of the student's knowledge.

The pronunciations here given are those of the Century Cyclopedia of Names. The first date indicates the year of birth, the second the year of death. All dates are A.D. unless the contrary is indicated by the sign —. The letter *c.* stands for *circa*, about, *b.* for *born*, *d.* for *died*. Numbers after the biographical note refer to pages in this work.

KEY. L. Latin, G. Greek, dim. diminutive, fem. feminine.

a *fat*,	ā *fate*,	ä *far*,	â *fall*,	ȧ *ask*,	ã *fare*,
e *met*,	ē *mete*,	ė *her*,	i *pin*,	ī *pine*,	o *not*,
ō *note*,	ö *move*,	ô *nor*,	u *tub*,	ū *mute*,	ů *pull*.
	ṅ French nasalizing *n*.			ċh German *ch*.	
	s̮ as in *leisure*.			t̮ as in *nature*.	

A single dot under a vowel indicates its abbreviation.

A double dot under a vowel indicates that the vowel approaches the short sound of *u*, as in *put*.

Ahmes (ä′mes). c. – 1700. Egyptian priest. Wrote the oldest extant work on mathematics 221

Anaxagoras (an-aks-ag′ō-ras). – 499, – 428. Greek philosopher and mathematician 225

Archimedes (är-ki-mē′dēz). – 287, – 212. Syracuse, Sicily. The greatest mathematician and physicist of antiquity . 87, 221, 353, 354

Aryabhatta (är-yä-bhä′ta). b. 476. One of the earliest Hindu mathematicians. Wrote on Algebra and Geometry 221

Bhaskara (bhäs′ka-ra). 12th cent. Hindu mathematician . . . 104

Brahmagupta (bräh-ma-göp′ta). b. 598. Hindu mathematician. One of the earliest Indian writers 143, 221

Carnot (kär-nō′), Lazare Nicholas Marguerite. 1753, 1823. French physicist and mathematician. Contributed to Modern Geometry 241, 242

Cavalieri (kä-vä-lē-ā′rē), Bonaventura. 1598, 1647. Prominent Italian mathematician 351

Ceulen (koi′len). Ludolph van. 1540, 1610. Dutch geometrician . 221

Ceva (chā′vä), Giovanni. 1648, c. 1737. Italian geometrician, 239, 241

Dase (dä′ze), Zacharias. 1824, 1861. Famous German computer . 221

Descartes (dā-kärt′), René. 1596, 1650. Eminent French mathematician, physicist, and philosopher. Founder of the science of Analytic Geometry 285

Euclid (ū′klid). c. – 300. Eminent writer on Geometry in the Alexandrian School, at Alexandria, Egypt. His "Elements," the first scientific text-book on the subject, is still the standard in the schools of England 76, 152, 208

Euler (oi′ler), Leonhard. 1707, 1783. Swiss. One of the greatest mathematicians of modern times 99, 108, 285, 289

Gauss (gous), Karl Friedrich. 1777, 1855. German. One of the greatest mathematicians of modern times 208, 212

Hero (hē′rō) of Alexandria. More properly Heron (hē′ron). c. – 110. Celebrated Greek surveyor and mechanician 221, 227

Hippocrates (hi-pok′ra-tēz) of Chios. b. c. – 470. Author of the first elementary text-book on Geometry 230

Jones (jōnz), William. 1675–1749. English teacher 221

Klein (klīn), Christian Felix. 1849. Professor at Göttingen . . 225

Leibnitz (līb′nits), Gottfried Wilhelm. 1646, 1716. Equally celebrated as a philosopher and a mathematician. One of the founders of the science of the Calculus 23, 182

Lindemann (lin′de-man), Ferdinand. b. 1852. German professor 225

TABLE OF ETYMOLOGIES.

This table includes such of the pronunciations and etymologies of the more common terms of Geometry as are of greatest value to the student. The equivalent foreign word is not always given, but rather the primitive root as being more helpful. The pronunciations and etymologies are those of the Century Dictionary. See Biographical Table, p. 372.

Abscissa (ab-sis′ä). L. cut off.

Acute (a-kūt′). L. *acutus*, sharp.

Adjacent (a-jā′sent). L. *ad*, to, + *jacere*, lie.

Angle (ang′gl). L. *angulus*, a corner, an angle; G. *ankylos*, bent.

Antecedent (an-tē-se′dent). L. *ante*, before, + *cedere*, go.

Bisect (bī-sekt′). L. *bi-*, two-, + *sectus*, cut.

Center (sen′tėr). L. *centrum*, center; G. *kentron*, from *kentein*, to prick.

Centroid (sen′troid). G. *kentron*, center, + *eidos*, form.

Chord (kôrd). G. *chorde*, string.

Circle (sir′kl). L. *circulus*, dim. of *circus* (G. *kirkos*), a ring.

Circumference (sėr-kum′fe-rens). L. *circum*, around (see Circle), + *ferre*, to bear.

Collinear (ko-lin′ē-är). L. *com-*, together, + *linea*, line.

Commensurable (ko-men′sū-ra-bl). L. *com-*, together, + *mensurare*, measure.

Complement (kom′plē-ment). L. *complementum*, that which fills, from *com-* (intensive) + *plere*, fill.

Concave (kon′-kāv). L. *com-* (intensive) + *cavus*, hollow.

Concentric (kon-sen′trik). L. *con-*, together, + *centrum*, center.

Concurrent (kon-kur′ent). L. *con-*, together, + *currere*, run.

Concyclic (kon-sik′lik). L. *con-*, together, + *cyclicus*, from G. *kyklikos*, from *kyklos*, a circle, related to *kyliein*, roll (compare Cylinder).

Congruent (kong′grö-ent). L. *congruere*, to agree.

Consequent (kon′sē-kwent). L. *con-*, together, + *sequi*, follow.

Constant (kon′stänt). L. *con-*, together, + *stare*, stand.

Converse (kon′vėrs). L. *con-*, together, + *vertere*, turn.

Convex (kon′veks). L. *convexus*, vaulted, from *con-*, together, + *vehere*, carry.

Corollary (kor′o-lā-ri). L. *corollarium*, a gift, money paid for a garland of flowers, from *corolla*, dim. of *corona*, a crown.

Cylinder (sil′in-dėr). G. *kylindros*, from *kyliein*, roll.

Decagon (dek′a̤-gon). G. *deka*, ten, + *gonia*, an angle.

Degree (dē-grē′). L. *de*, down, + *gradus*, step.

Diagonal (dī-ag′ō̤-na̤l). G. *dia*, through, + *gonia*, a corner, an angle.

Diameter (di-am′e-tėr). G. *dia*, through, + *metron*, a measure.

Dihedral (dī-hē′dra̤l). G. *di-*, two, + *hedra*, a seat.

Dimension (di-men′sho̤n). L. *dis-*, apart, + *metiri*, measure. See Measure.

Directrix (di-rek′triks). L. fem. of *director*, from *directus*, direct.

Dodecahedron (dō″dek-a̤-hē′dro̤n). G. *dodeka*, twelve, + *hedra*, a seat.

Equal (ē′kwa̤l). L. *æqualis*, equal, from *æquus*, plain.

Equiangular (ē-kwi-ang′gṳ̄-lär). L. *æquus*, equal, + *angulus*, angle.

Equilateral (ē-kwi-lat′e̤-ra̤l). L. *æquus*, equal, + *latus*, side.

Equivalent (ē̤-kwiv′a̤-le̤nt). L. *æquus*, equal, + *valere*, be strong.

Escribed (es-krībd′). L. *e*, out, + *scribere*, write.

Excess (ek-ses′). L. *ex*, out, + *cedere*, go; *i.e.* to pass beyond.

Frustum (frus′tum). L. a piece.

Genera̤trix (jen′e̤-rā-triks). L. fem. of *generator*, from *generare*, beget, from *genus*, a race.

Geometry (jē̤-om′e-tri). G. *ge*, the earth, + *metron*, a measure.

-gon, a termination, G. *gonia*, an angle.

Harmonic (här-mon′ik). G. *harmonia*, a concord, related to *harmos*, a joining. A line divided internally and externally in the ratio 2 : 1, is cut into segments representing 1, $\frac{2}{3}$, $\frac{1}{2}$. Pythagoras first discovered that a vibrating string stopped at half its length gave the octave of the original note, and stopped at two-thirds of its length gave the fifth. Hence the expression "harmonic division" of a line.

Hemisphere (hem′i-sfēr). G. *hemi-*, half, + *sphaira*, a sphere.

-hedron, a termination, G. *hedra*, a seat.

Hepta-, in combination, G. seven.

Hexa-, in combination, G. six.

Hexagram (hek′sa̤-gram). G. *hex*, six, + *gramma*, a line.

Hypotenuse (hī-pot′e-nūs). G. *hypo*, under, + *teinein*, stretch.

Inclination (in-kli-nā′sho̤n). L. *in*, on, + *clinare*, lean.

Incommensurable (in-kō̤-men′sṳ̄-ra̤-bl). L. *in-*, not, + *com-*, together, + *mensurare*, measure.

Infinity (in-fin′i̤-ti). L. *in-*, not, + *finitus*, bounded.

Inscribed (in-skrībd′). L. *in*, in, + *scribere*, write.

Isosceles (ī-sos′e-lēz). G. *isos*, equal, + *skelos*, leg.

Lateral (lat′e̤-ra̤l). L. *latus*, a side.

Locus (lō′kus). L. a place. Compare locality.

Lune (lūn). L. *luna*, the moon.

Major (mā′jo̤r). L. greater, comparative of *magnus*, great.

Maximum (mak′si-mum). L. greatest, superlative of *magnus*, great.

Mean (mēn). L. *medius*, middle.

Measure (mezh′ụ̄r). L. *mensura*, a measuring. See Dimension.

Median (mē′di-ạn). See Mean.

Mensuration (men-sụ̄-rā′shọn). See Measure.

Minimum (min′i-mum). L. least.

Minor (mī′nọr). L. less.

Nappe (nap). French, a cloth, sheet, surface.

Oblique (ọb-lēk′ or ọb-līk′). L. *ob*, before, + *liquis*, slanting.

Obtuse (ọb-tūs′). L. *obtusus*, blunt, from *ob*, upon, + *tundere*, strike.

Octo-, octa-, in combination, L. and G., eight.

Opposite (op′ọ̄-zit). L. *ob*, before, against, + *ponere*, put, set.

Ordinate (ôr′di-nạ̄t). L. *ordo* (*ordin-*), a row.

Orthocenter (ôr′thọ̄-sen-tėr). G. *ortho-*, straight, + *kentron*, center.

Orthogonal (ôr-thog′ọ̄-nạl). G. *orthos*, right, + *gonia*, an angle.

Parallel (par′ạ-lel). G. *para*, beside, + *allelon*, one another.

Parallelepiped (par-ạ-lel-e-pip′ed or -pī′ped). Gr. *parallelos*, parallel, + *epipedon*, a plane surface, from *epi*, on, + *pedon*, ground.

Parallelogram (par-ạ-lel′ọ̄-gram). G. *parallelos*, parallel, + *gramma*, line.

Pedal (ped′al or pē′dạl). L. *pedalis*, pertaining to the foot, from *pes* (*ped-*), foot.

Pencil (pen′sil). L. *penicillum*, a painters' pencil, a brush.

Perigon (per′i-gon). G. *peri*, around, + *gonia*, a corner, angle.

Perimeter (pẹ̄-rim′e-tėr). G. *peri*, around, + *metron*, measure.

Perpendicular (pėr-pẹn-dik′ụ̄-lạ̈r). L. *perpendiculum*, a plumb-line, from *per*, through, + *pendere*, hang.

Perspective (pėr-spek′tiv). L. *per*, through, + *specere*, see.

π (pī). Initial of G. *periphereia*, periphery, circumference.

Pole (pōl). G. *polos*, a pivot, hinge, axis, pole.

Polygon (pol′i-gon). G. *polys*, many, + *gonia*, corner, angle.

Polyhedron. (pol-i-hē′drọn) G. *polys*, many, + *hedra*, seat.

Postulate (pos′tụ̄-lāt). L. *postulatum*, a demand, from *poscere*, ask.

Prism (prizm). G. *prisma*, something sawed, from *priein*, saw.

Prismatoid (priz′mạ-toid). G. *prisma* (*t-*), + *eidos*, form.

Projection (prọ̄-jek′shọn). L. *pro*, forth. + *jacere*, throw.

Pyramid (pir′ạ-mid). G. *pyramis*, a pyramid, perhaps from Egyptian *pir-em-us*, the slanting edge of a pyramid.

Quadrant (kwod′rạnt). L. *quadran(t-)s*, a fourth part. See Quadrilateral.

Quadrilateral (kwod-ri-lat′ẹ-rạl). L. *quattuor* (*quadri-*), four, + *latus*, (*later-*), side.

Radius (rā′di-us). L. rod, spoke of a wheel.

Ratio (rā′shiō). L. a reckoning, calculation, from *reri*, think, estimate.

Reciprocal (rẹ̄-sip′rọ̄-kạl). L. *reciprocus*, returning, from *re-*, back, and *pro*, forward, with two adjective terminations.

Rectangle (rek′tang-gl). L. *rectus*,

right, + *angulus*, an angle. See Angle.

Rectilinear (rek-ti-lin′-ä̤r). L. *rectus*, right, + *linea*, a line.

Reflex (rē′fleks or re̤-fleks′). L. *re-*, back, + *flectere*, bend.

Regular (reg′ṳ̄-lä̤r). L. *regula*, a rule, from *regere*, rule, govern.

Rhombus (rom′bus). G. *rhombos*, a spinning top.

Scalene (skā̤-lēn′). G. *skalenos*, uneven, unequal; related to *skellos*, crooked-legged.

Secant (sē′kant). L. *secare*, cut, as also Sector, Section, Segment.

Segment (seg′ment). See Secant.

Semicircle (sem′i-sėr-kl). L. *semi-*, half, + *circulus*, circle.

Similar (sim′i-lä̤r). L. *similis*, like.

Solid (sol′id). L. *solidus*, firm, compact.

Sphere (sfēr). G. *sphaira*, a ball.

Square (skwār). L. *quadra*, a square, from *quattuor*, four.

Straight (strāt). Anglo-Saxon, *streht*, from *streccan*, stretch.

Subtend (sub-tend′). L. *sub*, under, + *tendere*, stretch.

Successive (suk-ses′iv). L. *sub*, under, + *cedere*, go.

Sum (sum). L. *summa*, highest part. Compare Summit.

Superposition (sū′pėr-pō-zish′o̤n). L. *super*, over, + *ponere*, lay.

Supplement (sup′le̤-me̤nt). L. *sub.*, under, + *plere*, fill; to fill up.

Surface (sėr-fā̤s). L. *superficies*, surface, from *super*, above, + *facies*, form, figure, face.

Symbol (sim′bo̤l). G. *symbolos*, a sign by which one infers something, from *syn*, together, + *ballein*, put.

Tangent (tan′je̤nt). L. *tangere*, touch.

Tetrahedron (tet-ra̤-hē′dro̤n). G. *tetra-*, four, + *hedra*, seat.

Theorem (thē′ō-rem). G. *theorema*, a sight, a principle contemplated.

Transversal (trȧns-vėr′sa̤l). L. *trans*, across, + *vertere*, turn.

Trapezium (trā̤-pē′zi-um). G. *trapezion*, a table, dim. of *trapeza*, a table, from *tetra*, four, + *pous*, foot.

Trapezoid (trā̤-pē′zoid). G. *trapeza*, table, + *eidos*, form.

Tri-, in composition, L. *tres* (*tri-*), G. *treis* (*tri-*), three. See Secant, -hedron, Angle, for meaning of trisect, trihedral, triangle.

Truncate (trung′kāt). L. *truncare*, cut off, from Old L. *troncus*, cut off, mutilated.

Unique (ṳ̄-nēk′). L. *unicus*, from *unus*, one.

Vertex (vėr′teks). L. *vertere*, turn.

Zone (zōn). G. *zone*, a girdle, belt.

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